Spectral Decomposition of Integral Operators for Optimal Interpolation in Hilbert Subspaces

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Abstract

The orthogonal projection associated to optimal interpolation in a Hilbert subspace is characterized by the spectral decomposition of problem adapted integral operators. We then propose a methodology to construct interpolators that take into account an infinite number of informations. As an application, we illustrate how boundary constraints can be enforced in Gaussian process models.

Keywords: Hilbert Subspaces, Optimal interpolation, Integral Operators, Spectral Decomposition, Infinite data set

1. Introduction

Let E be a separated topological real vector space and let E' be its topological dual. Let M be a linear subspace of E' and fixed $\varphi \in E$. An element $f \in E$ is said to be an interpolator of φ for M if

$$\forall e' \in M,\ \langle f, e' \rangle_{E,E'} = \langle \varphi, e' \rangle_{E,E'} .$$

The interpolation theory is consisted of the characterization and the construction of such interpolators. If E is also locally convex and quasi-complete, L. Schwartz theory of Hilbert subspaces of E (see [1]) gives an interesting and efficient framework for interpolation. Giving a Hilbert subspace $\mathcal{H}$ of E and $\varphi \in \mathcal{H} \subset E$, one can easily characterize the set of all interpolators, in $\mathcal{H}$, of φ for M from orthogonal projections in $\mathcal{H}$ (see section 2 for details). Let us also remark that interpolation in Hilbert subspaces and conditioning of Gaussian processes are intrinsically linked since the Hilbert structures appearing in each case are isometric (see [2, 3, 4]). Nevertheless, we will not address the more general problem of conditioning in this work.

In the most part of application cases, the set M is of finite dimension, which traduces the knowledge of a finite number of information about the unknown target element φ. To be more precise, we assume that we know the values $\langle \varphi, e'_i \rangle_{E,E'}$ for $1 \leqslant i \leqslant n$, with

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$e_i' \in E'$ (M is the linear subspace of E' spanned by the μ_i, $1 \leqslant i \leqslant n$). The interpolation problem is then associated with an orthogonal projection of $\mathcal{H}$ onto a linear subspace $\mathcal{H}_M$ of finite dimension. Due to its finite character, the construction of interpolators can in this case be numerically achieved.

In the present work, we are particularly interested in cases in which the subspace $\mathcal{H}_M$ is of infinite dimension (but separable), traducing the knowledge of an infinite number of (linearly independent) informations of type $\langle \varphi, e' \rangle_{E,E'}$, $e' \in M \subset E'$ (M is then necessarily of infinite dimension). For instance, if $\mathcal{H}$ is composed of real-valued functions on a subspace $\mathcal{X}$ of $\mathbb{R}^d$, such a situation would appear if one assumes that informations about the target function φ are known one continuous subset of $\mathcal{X}$ (for example, boundary values). Our aim is to give a way to numerically construct interpolators that take account of an infinite number of informations, or at least to approximate such interpolators (but without discretize the data set). In [5], a similar study has been accomplished for optimal interpolation in reproducing kernel Hilbert spaces (RKHS, see [6] and [1§9.]) of real-valued functions on a set $\mathcal{X}$ for information of the type "the values $\varphi(s)$ are known on a subspace $\mathcal{S}$ of $\mathcal{X}$" (Dirichlet boundary conditions, results for the conditioning of Gaussian process with continuous sample paths with such conditions can also be found in [7]). This paper is a generalization of the ideas appearing in [5] to L. Schwartz Hilbert subspaces and for general sets $\mathcal{H}_M$ (see remark 3.7).

The first part (section 2) of this article is devoted to the description of the theoretical background of interpolation in Hilbert subspaces. In section 3, we present interpolation-adapted integral operators and discuss their properties. We next use the spectral decomposition of the those operators in order to construct interpolators. In section 5, we consider the case of a finite number of informations. We prove that the spectral representation formula for the optimal interpolation given in section 4 is equivalent to its usual expression. We conclude with an example of application, we consider a Hilbert subspace composed of continuously derivable real-valued functions on $\mathbb{R}^2$ and show how one can enforce constraints to the normal derivatives and the values of interpolators on a circle. We illustrate our results with an example of their potential application in Gaussian process models (kriging models).

2. Theoretical Background : Optimal interpolation in Hilbert subspaces

Let E be a quasi-complete, locally convex, separated topological real-vector space (see for instance [8]) and denote E' its topological dual space. Let $\mathcal{H}$ be a Hilbert subspace of E, we use the notation $\mathcal{H} \in Hilb(E)$. We denote $T_{\mathcal{H}}$ the Hilbert kernel naturally associated with $\mathcal{H}$. We remind that $T_{\mathcal{H}} : E' \to \mathcal{H} \subset E$ verify

$$\forall h \in \mathcal{H},\ \forall e' \in E',\ \langle h, e' \rangle_{E,E'} = (h|T_{\mathcal{H}}e')_{\mathcal{H}}\,,$$

where $(\cdot|\cdot)_{\mathcal{H}}$ is the inner product of $\mathcal{H}$.

Let M be a linear subspace of E', we pose

$$M^0 = \Big\{ e \in E \ :\ \forall e' \in M,\ \langle e, e' \rangle_{E,E'} = 0 \Big\}.$$

We define $\mathcal{H}_0 = M^0 \cap \mathcal{H} = T_{\mathcal{H}}(M)^{\perp}$, where $T_{\mathcal{H}}(M)^{\perp}$ denotes the orthogonal, in $\mathcal{H}$, of $T_{\mathcal{H}}(M)$. Hence, for a fixed $\varphi \in \mathcal{H}$,

$$\varphi + \left(M^0 \cap \mathcal{H}\right)$$

is the set of all interpolators, in $\mathcal{H}$, of φ for M.

By definition, $\varphi + \left(M^0 \cap \mathcal{H}\right)$ is a non-empty, closed affine subspace of $\mathcal{H}$, hence convex. Thus $\varphi + \left(M^0 \cap \mathcal{H}\right)$ admits a minimal norm element, which we denote $h_{\varphi,M}$ and call minimal norm interpolator, or optimal interpolator. $h_{\varphi,M}$ is the orthogonal projection of 0 onto $\varphi + \left(M^0 \cap \mathcal{H}\right)$.

By definition of the orthogonal projection, $h_{\varphi,M}$ is orthogonal to $\mathcal{H}_0$, i.e.

$$h_{\varphi,M} \in \mathcal{H}_0^{\perp} = \left(T_{\mathcal{H}}\left(M\right)^{\perp}\right)^{\perp} = \overline{T_{\mathcal{H}}\left(M\right)}^{\mathcal{H}} = \mathcal{H}_M,$$

with $\mathcal{H}_M$ the closure, in $\mathcal{H}$, of the linear space spanned by $T_{\mathcal{H}}e'$, $e' \in M$.

Because of the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_M$, $h_{\varphi,M}$ is the only interpolator, in $\mathcal{H}_M$, of φ for M.

Finally, let $P_{\mathcal{H}_M}$ be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$. $\varphi - P_{\mathcal{H}_M}\left[\varphi\right]$ is orthogonal to $\mathcal{H}_M$, thus $\varphi - P_{\mathcal{H}_M}\left[\varphi\right] \in \mathcal{H}_0$, i.e. $P_{\mathcal{H}_M}\left[\varphi\right]$ interpolates φ for M, which means that

$$h_{\varphi,M} = P_{\mathcal{H}_M}\left[\varphi\right].$$

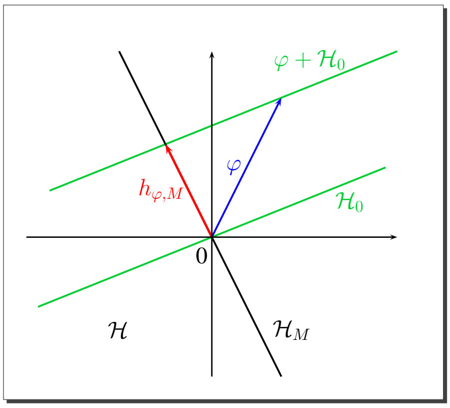

Figure 1: Schematic representation of optimal interpolation in a Hilbert subspace.

In practice, the set of all interpolators is characterized by $h_{\varphi,M}$ and the kernel $T_{\mathcal{H}_0}$ (the knowledge of those two objects allows the construction of all the interpolators, in $\mathcal{H}$, of φ for M).

The Hilbert kernel $T_{\mathcal{H}_M}$ of the Hilbert subspace $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$ is linked with $T_{\mathcal{H}}$ by the relation

$$T_{\mathcal{H}_M} = P_{\mathcal{H}_M} T_{\mathcal{H}},$$

this result being true for all closed linear subspace of $\mathcal{H}$. Hence, the knowledge of $T_{\mathcal{H}_M}$ defines the orthogonal projection $P_{\mathcal{H}_M}$ and reciprocally.

At least, note that if $\{h_i,\, i \in I\}$ is a Hilbert basis of $\mathcal{H} \in Hilb(E)$, the Hilbert kernel $T_{\mathcal{H}}$ of $\mathcal{H}$ can be written under the form :

$$T_{\mathcal{H}} = \sum_{i \in I} h_i \otimes h_i, \text{ i.e. } \forall e' \text{ and } f' \in E',\ \left\langle T_{\mathcal{H}}e', f'\right\rangle_{E,E'} = \sum_{i \in I} \left\langle h_i, e'\right\rangle_{E,E'} \left\langle h_i, f'\right\rangle_{E,E'}.$$

Remark 2.1 : The following result is specially motivated by the example treated in section 6.2. It shows how to solve the interpolation problem associated with a linear subspace M of E' by splitting M in several linear subspaces.

Let $M1$ and $M2$ be two linear subspaces of E'. We introduce

$$\mathcal{H}_{M1} = \overline{T_{\mathcal{H}}(M1)}^{\mathcal{H}} \text{ and } \mathcal{H}_{01} = \mathcal{H}_{M1}^{\perp}.$$

Let us denote $P_{\mathcal{H}_{M1}}$ and $P_{\mathcal{H}_{01}}$ the orthogonal projection of $\mathcal{H}$ onto respectively $\mathcal{H}_{M1}$ and $\mathcal{H}_{01}$ and let $T_{\mathcal{H}_{01}} = P_{\mathcal{H}_{01}} T_{\mathcal{H}}$ be the Hilbert kernel of $\mathcal{H}_{01}$. Finally, let

$$\mathcal{H}_{01,M2} = \overline{T_{\mathcal{H}_{01}}(M2)}^{\mathcal{H}}.$$

Then, for $\varphi \in \mathcal{H}$, the optimal interpolator of φ for $M = M1 + M2$ is given by

$$P_{\mathcal{H}_M}[\varphi] = P_{\mathcal{H}_{M1}}[\varphi] + P_{\mathcal{H}_{01,M2}}[\varphi - P_{\mathcal{H}_{M1}}[\varphi]],$$

and $T_{\mathcal{H}_0} = P_{\mathcal{H}_{01,02}} P_{\mathcal{H}_{01}} T_{\mathcal{H}}$, with $P_{\mathcal{H}_{01,M2}}$ the orthogonal projection of $\mathcal{H}_{01}$ onto $\mathcal{H}_{01,M2}$, and $\mathcal{H}_{01,02}$ the orthogonal of $\mathcal{H}_{01,M2}$ in $\mathcal{H}_{01}$. ■

3. Problem Adapted Integral Operator

Let $\mathcal{H} \in Hilb(E)$ and let M be a linear subspace of E'. We use the same notations and definitions than in section 2.

In this section, we will show how, from compact integral operators spectral decomposition, one can build a Hilbert basis of the linear subspace $\mathcal{H}_M$ of $\mathcal{H}$ (note that such a $\mathcal{H}_M$ would then be necessarily separable, see for instance corollary 3.2).

Let $\mathcal{S}$ be a general set endowed with a σ-algebra and let ν be a σ-finite measure on $\mathcal{S}$. Let $\gamma : \mathcal{S} \to E'$ be such that, for all $h \in \mathcal{H}$, the function $\mathfrak{f}_h : s \in \mathcal{S} \mapsto \langle h, \gamma s\rangle_{E,E'} \in \mathbb{R}$ is measurable and

$$\forall h \in \mathcal{H}, \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'}^2 \, d\nu(s) < +\infty. \tag{3.1}$$

We assume that the linear space spanned by the γs, $s \in \mathcal{S}$ is M (i.e. $M = span\{\gamma(\mathcal{S})\}$).

Let $L^2(\mathcal{S}, \nu)$ be the Hilbert space of square-integrable real-valued functions with respect to ν ($L^2(\mathcal{S}, \nu)$ is obviously a quotient space), note that $L^2(\mathcal{S}, \nu)$ has not to be necessarily separable. Let $(\cdot|\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ be its inner product and norm. We remind that

$$(f|g)_{L^2} = \int_{\mathcal{S}} f(s)g(s)d\nu(s).$$

Equation (3.1) means that, for all $h \in \mathcal{H}$, $\mathfrak{f}_h \in L^2(\mathcal{S}, \nu)$. We then define the application $\mathfrak{F} : \mathcal{H} \to L^2(\mathcal{S}, \nu)$, $h \mapsto \mathfrak{f}_h$. Because $M = span\{\gamma(\mathcal{S})\}$, we have $\mathfrak{F}(\mathcal{H}_0) = 0$, i.e. in terms of set, $\mathfrak{F}(\mathcal{H}) = \mathfrak{F}(\mathcal{H}_M)$.

We add the following hypotheses :

H-i. the application $(s,t) \in \mathcal{S} \times \mathcal{S} \mapsto \langle T_{\mathcal{H}}\gamma s, \gamma t\rangle_{E,E'} = (T_{\mathcal{H}}\gamma s | T_{\mathcal{H}}\gamma t)_{\mathcal{H}}$ is measurable,

H-ii. $N = \int_{\mathcal{S}} \|T_{\mathcal{H}}\gamma s\|_{\mathcal{H}}^2 \, d\nu(s) < +\infty$, H-iii. for $h \in \mathcal{H}$, $\|\mathfrak{f}_h\|_{L^2} = 0$ if and only if $h \in \mathcal{H}_0$.

Remark 3.1 : Hypothesis H-i and H-ii imply that, for all $h \in \mathcal{H}$,

$$\|\mathfrak{f}_h\|_{L^2}^2 = \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'}^2 \, d\nu(s) = \int_{\mathcal{S}} (h|T_{\mathcal{H}}\gamma s)_{\mathcal{H}}^2 \, d\nu(s) \leqslant N \|h\|_{\mathcal{H}}^2 . \tag{3.2}$$

Furthermore, from Cauchy-Schwarz inequality :

$$\begin{aligned} \int_{\mathcal{S}}\int_{\mathcal{S}} \langle T_{\mathcal{H}}\gamma s, \gamma t\rangle_{E,E'}^2 \, d\nu(s)d\nu(t) &= \int_{\mathcal{S}}\int_{\mathcal{S}} (T_{\mathcal{H}}\gamma s|T_{\mathcal{H}}\gamma t)_{\mathcal{H}}^2 \, d\nu(s)d\nu(t) \\ &\leqslant \int_{\mathcal{S}}\int_{\mathcal{S}} \|T_{\mathcal{H}}\gamma s\|_{\mathcal{H}}^2 \|T_{\mathcal{H}}\gamma t\|_{\mathcal{H}}^2 \, d\nu(s)d\nu(t) = N^2. \end{aligned}$$

We will call $\|T_{\mathcal{H}}\|_{\gamma,\nu}^2$ this quantity, i.e.

$$\|T_{\mathcal{H}}\|_{\gamma,\nu}^2 = \int_{\mathcal{S}}\int_{\mathcal{S}} \langle T_{\mathcal{H}}\gamma s, \gamma t\rangle_{E,E'}^2 \, d\nu(s)d\nu(t) \quad \left(\leqslant N^2\right). \tag{3.3}$$

Hypothesis H-iii will be realized for example if $\mathcal{S}$ is a topological space (endowed with its Borel σ-algebra), if γ is continuous for the weak topology of E' and if $supp(\nu) = \mathcal{S}$ (with $supp(\nu)$ the support of ν).

Finally, note that H-iii and equation (3.2) imply that $\mathfrak{F} : \mathcal{H}_M \to L^2(\mathcal{S},\nu)$ is a continuous injection. ■

We endowed $\mathfrak{F}(\mathcal{H}) = \mathfrak{F}(\mathcal{H}_M)$ of the following inner-product,

$$\forall f \text{ and } g \in \mathcal{H}_M, \ (\mathfrak{f}_h|\mathfrak{f}_g)_{\mathfrak{F}(\mathcal{H})} = (h|g)_{\mathcal{H}} . \tag{3.4}$$

From H-iii, $\mathfrak{F}(\mathcal{H}) = \mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ is a Hilbert space and, by construction, it is isometric to $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$. In addition, from equation (3.2), the inclusion of $\mathfrak{F}(\mathcal{H}_M)$ into $L^2(\mathcal{S},\nu)$ is continuous (i.e. $\mathfrak{F}(\mathcal{H}_M) \in Hilb\left(L^2(\mathcal{S},\nu)\right)$, see remark 3.2).

We now introduce the operator $\mathcal{L}_\nu$, which is at this moment correctly define from $L^2(\mathcal{S},\nu)$ onto $\mathbb{R}^{\mathcal{S}}$ (with $\mathbb{R}^{\mathcal{S}}$ the space of real-valued functions on $\mathcal{S}$). We pose :

$$\forall s \in \mathcal{S} \text{ and } \forall f \in L^2(\mathcal{S},\nu), \ \mathcal{L}_\nu[f](s) = \int_{\mathcal{S}} (T_{\mathcal{H}}\gamma t|T_{\mathcal{H}}\gamma s)_{\mathcal{H}} f(t)d\nu(t).$$

Proposition 3.1

The linear space spanned by the $\mathcal{L}_\nu[f]$, $f \in L^2(\mathcal{S},\nu)$ is dense in $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$. Furthermore, for all $f \in L^2(\mathcal{S},\nu)$ and $h \in \mathcal{H}$,

$$(\mathfrak{f}_h|\mathcal{L}_\nu[f])_{\mathfrak{F}(\mathcal{H})} = (\mathfrak{f}_h|f)_{L^2} . \tag{3.5}$$

Proof: For a fixed $f \in L^2(\mathcal{S},\nu)$, we define the linear application $I_f : L^2(\mathcal{S},\nu) \to \mathbb{R}$ such that, for $g \in L^2(\mathcal{S},\nu)$, $I_f[g] = (f|g)_{L^2}$. Thus, for $h \in \mathcal{H}$,

$$|I_f[\mathfrak{f}_h]| \leqslant \|f\|_{L^2} \|\mathfrak{f}_h\|_{L^2} \leqslant \|f\|_{L^2} \sqrt{N} \|h\|_{\mathcal{H}} \ \left(= \|f\|_{L^2} \sqrt{N} \|\mathfrak{f}_h\|_{\mathfrak{F}(\mathcal{H})}\right).$$

I_f is a linear and continuous application from the Hilbert $\mathfrak{F}(\mathcal{H})$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ onto $\mathbb{R}$, thus, for all $h \in \mathcal{H}$ there exists a unique $R_f \in \mathcal{H}_M$ such that

$$I_f[\mathfrak{f}_h] = \left(\mathfrak{f}_h|\mathfrak{f}_{R_f}\right)_{\mathfrak{F}(\mathcal{H})} = (h|R_f)_{\mathcal{H}}.$$

Nevertheless, for all $s \in \mathcal{S}$,

$$\begin{aligned} \mathfrak{f}_{R_f}(s) &= \langle R_f, \gamma s\rangle_{E,E'} = (R_f|T_{\mathcal{H}}\gamma s)_{\mathcal{H}} = (f|\mathfrak{f}_{T_{\mathcal{H}}\gamma s})_{L^2} \\ &= \int_{\mathcal{S}} (T_{\mathcal{H}}\gamma t|T_{\mathcal{H}}\gamma s)_{\mathcal{H}}\, f(t)d\nu(t) = \mathcal{L}_\nu[f](s), \end{aligned}$$

which proves that $\mathcal{L}_\nu[f] \in \mathfrak{F}(\mathcal{H}_M)$. To obtain equation (3.5), we just have to remark that for $f \in L^2(\mathcal{S},\nu)$ and $h \in \mathcal{H}$,

$$(\mathfrak{f}_h|\mathcal{L}_\nu[f])_{\mathfrak{F}(\mathcal{H})} = (h|R_f)_{\mathcal{H}} = (\mathfrak{f}_h|f)_{L^2}$$

Finally, if $h \in \mathcal{H}_M$ verify

$$\forall f \in L^2(\mathcal{S},\nu),\ (\mathfrak{f}_h|\mathcal{L}_\nu[f])_{\mathfrak{F}(\mathcal{H})} = 0,$$

then, from hypothesis H-iii, $h = 0$. ■

Remark 3.2 : Proposition 3.1 is similar to

Corollaire 3.1

$\mathcal{L}_\nu$ is the Hilbert kernel of the Hilbert subspace $\mathfrak{F}(\mathcal{H}_M)$ of $L^2(\mathcal{S},\nu)$.

In fact, $L^2(\mathcal{S},\nu)$ being a Hilbert space, one can identify its topological dual to itself.

$$\mathcal{L}_\nu : \left(L^2(\mathcal{S},\nu)\right)' = L^2(\mathcal{S},\nu) \to \mathfrak{F}(\mathcal{H}_M) \subset L^2(\mathcal{S},\nu)$$

verify equation 3.5, which is exactly the characterization of the Hilbert kernel associated to $\mathfrak{F}(\mathcal{H}_M) \in Hilb\left(L^2(\mathcal{S},\nu)\right)$.

In a same way, the application $f \mapsto R_f$ appearing in the proof of proposition 3.1 could be seen as the Hilbert kernel of $\mathcal{H}_M$ relatively to $L^2(\mathcal{S},\nu)$ in the sense that, for all $h \in \mathcal{H}_M$ and $f \in L^2(\mathcal{S},\nu)$, $(f|\mathfrak{f}_h)_{L^2} = (h|R_f)_{\mathcal{H}}$. ■

Proposition 3.1 authorizes us to see $\mathcal{L}_\nu$ as an operator from $L^2(\mathcal{S},\nu)$ onto $L^2(\mathcal{S},\nu)$.

The Hilbert kernel $T_{\mathcal{H}}$ and the application γ define a symmetric and positive-definite kernel $\mathcal{K}(\cdot,\cdot)$ on $\mathcal{S}\times\mathcal{S}$:

$$\forall (s,t) \in \mathcal{S}\times\mathcal{S},\ \mathcal{K}(s,t) = (T_{\mathcal{H}}\gamma t|T_{\mathcal{H}}\gamma s)_{\mathcal{H}}. \tag{3.6}$$

From equation (3.3), $\mathcal{K}(\cdot,\cdot) \in L^2(\mathcal{S}\times\mathcal{S},\nu\times\nu)$ (we have $\|\mathcal{K}\|_{L^2} = \|T_{\mathcal{H}}\|_{\gamma,\nu}$), hence $\mathcal{L}_\nu$ can be seen as a classic Hilbert-Schmidt integral operator and is therefore compact (see for instance [9§10]).

Because $\mathcal{K}(\cdot,\cdot)$ is symmetric, $\mathcal{L}_\nu$ is self-adjoint ; further $\mathcal{L}_\nu$ is positive, apply for example equation (3.5). One also can note that $\mathcal{L}_\nu$ is continuous :

$$\forall f \in L^2(\mathcal{S},\nu),\ \|\mathcal{L}_\nu[f]\|^2_{L^2} \leqslant \|T_{\mathcal{H}}\|^2_{\gamma,\nu}\, \|f\|^2_{L^2}.$$

So $\mathcal{L}_\nu : L^2(\mathcal{S},\nu) \to L^2(\mathcal{S},\nu)$ is diagonalizable and its eigenvalues are positive.

Remark 3.3 : The fact that $\mathcal{L}_\nu : L^2(\mathcal{S},\nu) \to L^2(\mathcal{S},\nu)$ is a symmetric, positive and continuous operator can also be seen as a trivial consequence of corollary 3.1 and [1]. ■

We denote λ_i those eigenvalues and $\widetilde{\phi}_i \in L^2(\mathcal{S},\nu)$ their associated eigenfunctions, $i \in \mathbb{I}$. We remind that $\left\{\widetilde{\phi}_i,\ i \in \mathbb{I}\right\}$ form a orthonormal basis of $L^2(\mathcal{S},\nu)$. Finally, we will denote by $\{\lambda_n,\ n \in \mathbb{I}_+\}$ the countable set (i.e $\mathbb{I}_+ \subset \mathbb{N}$) off all **strictly positive** eigenvalues with multiplicity.

Equation (3.5) implies that, for all $n \in \mathbb{I}_+$,

$$\forall h \in \mathcal{H},\ \left(\widetilde{\phi}_n | \mathfrak{f}_h\right)_{\mathfrak{F}(\mathcal{H})} = \frac{1}{\lambda_n}\left(\mathcal{L}_\nu\left[\widetilde{\phi}_n\right] | \mathfrak{f}_h\right)_{\mathfrak{F}(\mathcal{H})} = \frac{1}{\lambda_n}\left(\widetilde{\phi}_n | \mathfrak{f}_h\right)_{L^2}. \tag{3.7}$$

Proposition 3.2

$\left\{\sqrt{\lambda_n}\widetilde{\phi}_n,\ n \in \mathbb{I}_+\right\}$ is a Hilbert basis of $\mathfrak{F}(\mathcal{H})$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$.

Proof : The fact that $\left\{\sqrt{\lambda_n}\widetilde{\phi}_n\right\}$ is an orthonormal system of $\mathfrak{F}(\mathcal{H})$ is consequence of equation (3.7). We just have to show that *span* $\left\{\sqrt{\lambda_n}\widetilde{\phi}_n\right\}$ is dense in $\mathfrak{F}(\mathcal{H})$ which is a consequence of proposition 3.1. ■

Remark 3.4 : The two spaces $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$, $(\cdot|\cdot)_{L^2}$ and $\mathfrak{F}(\mathcal{H})$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ are isometric. The isometry is given by :

$$\forall n \in \mathbb{I}_+,\ \widetilde{\phi}_n \leftrightarrow \sqrt{\lambda_n}\widetilde{\phi}_n.$$

In fact, this isometry is the restriction, at $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$, of the square-root of $\mathcal{L}_\nu : L^2(\mathcal{S},\nu) \to L^2(\mathcal{S},\nu)$,

$$\mathcal{L}_\nu^{\frac{1}{2}}\left[\sum_{i\in\mathbb{I}} \alpha_i \widetilde{\phi}_i\right] = \sum_{i\in\mathbb{I}_+} \alpha_i \sqrt{\lambda_i}\widetilde{\phi}_i \text{ with,}$$

for $i \in \mathbb{I}\backslash\mathbb{I}_+$, i.e. $\lambda_i = 0$, $\mathcal{L}_\nu^{\frac{1}{2}}\left[\widetilde{\phi}_i\right] = 0$. Obviously, $\mathcal{L}_\nu = \mathcal{L}_\nu^{\frac{1}{2}} \circ \mathcal{L}_\nu^{\frac{1}{2}}$. ■

As we have seen, $\mathfrak{F} : \mathcal{H}_M \to \mathfrak{F}(\mathcal{H}_M)$ define an isometry between the two Hilbert spaces $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{\mathfrak{F}(\mathcal{H})}$ and $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$. We then introduce $\mathfrak{F}^{-1} : \mathfrak{F}(\mathcal{H}_M) \to \mathcal{H}_M$, the inverse isometry of $\mathfrak{F}$.

For $n \in \mathbb{I}_+$, we denote $\phi_n = \mathfrak{F}^{-1}\widetilde{\phi}_n \in \mathcal{H}_M$. We call ϕ_n the **regularized eigenfunctions** associated with the eigenvalue λ_n, we will also say that ϕ_n is the *regularization* of $\widetilde{\phi}_n \in L^2(\mathcal{S},\nu)$.

Corollaire 3.2

$\left\{\sqrt{\lambda_n}\phi_n,\ n \in \mathbb{I}_+\right\}$ is a Hilbert basis of $\mathcal{H}_M$ $(\cdot|\cdot)_{\mathcal{H}}$ (and $\mathcal{H}_M$ is obviously separable).

Remark 3.5 : The application $\mathfrak{F}^{-1}$ is the generalization of the notion of regularization (or regularized eigenfunctions) that can be found in [5]. ■

The definition $\phi_n = \mathfrak{F}^{-1}\widetilde{\phi}_n$, $n \in \mathbb{I}_+$ correctly characterizes the regularized eigenfunctions as elements of $\mathcal{H}_M$. Nevertheless, in practice, we will need explicit formulas for the ϕ_n (in order to compute them for example). So, we consider the application $f \mapsto R_f$ appearing in the proof of proposition 3.1 and which we now denote $\mathfrak{F}^{-1}\mathcal{L}_\nu$ (coherent notation). We remind that, $\forall f \in L^2(\mathcal{S}, \nu)$ and $\forall s \in \mathcal{S}$,

$$\left\langle \mathfrak{F}^{-1}\mathcal{L}_\nu[f], \gamma s\right\rangle_{E,E'} = \int_{\mathcal{S}} (T_{\mathcal{H}}\gamma t | T_{\mathcal{H}}\gamma s)_{\mathcal{H}} f(t) d\nu(t). \tag{3.8}$$

From this equation, one can write $\mathfrak{F}^{-1}\mathcal{L}_\nu[f]$, $f \in L^2(\mathcal{S}, \nu)$, under a vectorial integral form (see [10]),

$$\mathfrak{F}^{-1}\mathcal{L}_\nu[f] = \int_{\mathcal{S}} f(t) T_{\mathcal{H}}\gamma t \, d\nu(t) \in \mathcal{H}_M. \tag{3.9}$$

Finally, we obtain (by using the linearity of $\mathfrak{F}^{-1}$), for all $n \in \mathbb{I}_+$,

$$\phi_n = \mathfrak{F}^{-1}\widetilde{\phi}_n = \frac{1}{\lambda_n}\mathfrak{F}^{-1}\mathcal{L}_\nu\left[\widetilde{\phi}_n\right] = \frac{1}{\lambda_n}\int_{\mathcal{S}} \widetilde{\phi}_n(t) T_{\mathcal{H}}\gamma t \, d\nu(t), \tag{3.10}$$

the expression (3.10) having to be understood in the sense of equation (3.8).

Remark 3.6 (an equivalent way to proceed) : Instead of transporting the Hilbert structure of $\mathcal{H}_M$ onto the subspace $\mathfrak{F}(\mathcal{H}_M)$ of $L^2(\mathcal{S}, \nu)$, one can remark that equation (3.1) allows us to define on $\mathcal{H}$ the symmetric and positive bilinear form

$$\forall h \text{ and } g \in \mathcal{H}, \; (h|g)_{\gamma,\nu} = \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'} \langle g, \gamma s\rangle_{E,E'} \, d\nu(s). \tag{3.11}$$

We denote $\|h\|^2_{\gamma,\nu} = (h|h)_{\gamma,\nu}$.

H-iii, implies that the null space of $(\cdot|\cdot)_{\gamma,\nu}$ is $\mathcal{H}_0$, i.e. for $h \in \mathcal{H}$, $\|h\|_{\gamma,\nu} = 0$ if and only if $h \in \mathcal{H}_0$. We have, for all $h \in \mathcal{H}$, $\|h\|_{\gamma,\nu} = \|\mathfrak{f}_h\|_{L^2}$ and $\mathcal{H}_M$, $(\cdot|\cdot)_{\gamma,\nu}$ is a pre-Hilbert space isometric to $\mathfrak{F}(\mathcal{H}_M)$, $(\cdot|\cdot)_{L^2}$, the isometry being the application $\mathfrak{F}$. The completed $\overline{\mathcal{H}_M}^{\gamma,\nu}$ of $\mathcal{H}_M$ for $\|\cdot\|_{\gamma,\nu}$ is then isometric, in terms of Hilbert structure, to the space $\overline{\mathfrak{F}(\mathcal{H})}^{L^2}$ of remark 3.4.

We next define the analogous L_ν of $\mathcal{L}_\nu$, firstly on $\mathcal{H}_M$, by

$$\forall h \in \mathcal{H}_M, \; \forall s \in \mathcal{S}, \; \langle L_\nu[h], \gamma s\rangle_{E,E'} = \mathcal{L}_\nu\left[\mathfrak{f}_h\right](s), \text{ or,} \tag{3.12}$$

equivalently, by $L_\nu[h] = \mathfrak{F}^{-1}\mathcal{L}_\nu\left[\mathfrak{f}_h\right]$. As mentioned in equation (3.9), L_ν can be written under the vectorial integral form

$$\forall h \in \mathcal{H}_M, \; L_\nu[h] = \int_{\mathcal{S}} \langle h, \gamma s\rangle_{E,E'} T_{\mathcal{H}}\gamma s \, d\nu(s). \tag{3.13}$$

L_ν is next extended to $\overline{\mathcal{H}_M}^{\gamma,\nu}$ by continuity. Then, one can proof that L_ν is a Hilbert-Schmidt operator on $\overline{\mathcal{H}_M}^{\gamma,\nu}$, $(\cdot|\cdot)_{\gamma,\nu}$. L_ν is symmetric and positive definite, its eigenvalues are λ_n, $n \in \mathbb{I}_+$ and each ones are associated with the eigenfunction ϕ_n.

We finally obtain the same Hilbert basis $\left\{\sqrt{\lambda_n}\phi_n,\ n\in\mathbb{I}_+\right\}$ of $\mathcal{H}_M\ (\cdot|\cdot)_{\mathcal{H}}$ than in corollary 3.2.

Note that one can also directly proof that L_ν, seen as an operator from $\mathcal{H}$ onto $\mathcal{H}_M$, is a positive and symmetric Hilbert-Schmidt operator. The eigenspace associated with the null eigenvalues is $\mathcal{H}_0$ and $\sqrt{\lambda_n}\phi_n,\ n\in\mathbb{I}_+$ are the (normed) eigenfunctions associated with the eigenvalues λ_n.

As an illustration, we will use this point of view in section 5. ■

Remark 3.7 (link with preceding works) : In [5] and [7], $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS, see [6]) of real-valued function defined on a general set $\mathcal{X}$, i.e. $\mathcal{H}$ is a Hilbert subspace of $E=\mathbb{R}^{\mathcal{X}}$ with $\mathbb{R}^{\mathcal{X}}$ the space of real valued functions on $\mathcal{X}$ endowed with the simple convergence topology (see [1§.9]). E' is the linear space spanned by the δ_x, $x\in\mathcal{X}$ where δ_x is the Dirac measure at x, i.e. for $f\in\mathbb{R}^{\mathcal{X}}$, $f(x)=\langle f,\delta_x\rangle_{E,E'}$. We remind that the reproducing kernel $K(\cdot,\cdot)$ of $\mathcal{H}$ is linked with its Hilbert kernel $T_{\mathcal{H}}$ by the relation

$$\forall x \text{ and } y\in\mathcal{X},\ K(x,y)=\langle T_{\mathcal{H}}\delta_x,\delta_y\rangle_{E,E'}\,.$$

For a subset $\mathcal{S}\subset\mathcal{X}$ endowed with a σ-algebra and a σ-finite measure ν on $\mathcal{S}$, the studied integral operator is

$$\forall f\in L^2(\mathcal{S},\nu),\ \mathcal{L}_\nu[f](s)=\int_{\mathcal{S}}K(s,t)f(t)d\nu(t).$$

Hence, the implicitly considered application γ is $\gamma:\mathcal{S}\to E'$, $s\mapsto\delta_s$ (and M is the linear subspace of E' spanned by the δ_s, $s\in\mathcal{S}$). So, the integral operators defined in section 3 generalize the ones considered in [5] and [7]. ■

4. Representation and Approximation of the Optimal Interpolator

Let $\mathcal{H}\in Hilb(E)$ and let M be a linear subspace of E'. In order to apply section 3, we assume that M is such that $\mathcal{H}_M$ is separable. We then consider the Hilbert basis $\left\{\sqrt{\lambda_n}\phi_n,\ n\in\mathbb{I}_+\right\}$ of $\mathcal{H}_M$, $(\cdot|\cdot)_{\mathcal{H}}$ which is defined in corollary 3.2. Knowing a Hilbert basis of $\mathcal{H}_M$, one can easily expressed, in terms of this basis, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_M$, i.e. expressed, for a fixed $\varphi\in\mathcal{H}$, the optimal interpolator of φ for M (see section 2). Finally, the property of the basis $\left\{\sqrt{\lambda_n}\phi_n,\ n\in\mathbb{I}_+\right\}$ allows us to give a formula for the optimal interpolator $P_{\mathcal{H}_M}[\varphi]$ which explicitly depends of the values $\langle\varphi,\gamma s\rangle_{E,E'}$, $s\in\mathcal{S}$, i.e. of the $\langle\varphi,e'\rangle_{E,E'}$, $e'\in M$.

Theorem 4.1

Let $\varphi\in\mathcal{H}$ and let M be a linear subspace of E' such that $\mathcal{H}_M$ is separable. Let us consider the Hilbert basis $\left\{\sqrt{\lambda_n}\phi_n,\ n\in\mathbb{I}_+\right\}$ of $\mathcal{H}_M$ defined in corollary 3.2, then we have

$$P_{\mathcal{H}_M}[\varphi]=\sum_{n\in\mathbb{I}_+}\phi_n\int_{\mathcal{S}}\langle\phi_n,\gamma s\rangle_{E,E'}\langle\varphi,\gamma s\rangle_{E,E'}\,d\nu(s). \tag{4.1}$$

Proof : It is a simple consequence of equation (3.7) :

$$\begin{aligned} P_{\mathcal{H}_M}[\varphi] &= \sum_n \sqrt{\lambda_n}\phi_n \left(\sqrt{\lambda_n}\phi_n|\varphi\right)_{\mathcal{H}} = \sum_n \lambda_n \phi_n \left(\widetilde{\phi}_n|\mathfrak{f}_\varphi\right)_{\mathfrak{F}(\mathcal{H})} \\ &= \sum_n \phi_n \left(\widetilde{\phi}_n|\mathfrak{f}_\varphi\right)_{L^2}. \end{aligned}$$

■

Remark that, by construction, the sum appearing in equation (4.1) converges in $\mathcal{H}$. Since $\mathcal{H} \in Hilb(E)$, it also converges for the initial topology of E and for its weak topology $\sigma(E, E')$. Then, in particular, for all $e' \in E'$,

$$\langle P_{\mathcal{H}_M}[\varphi], e'\rangle_{E,E'} = \sum_{n\in\mathbb{I}_+} \langle \phi_n, e'\rangle_{E,E'} \int_{\mathcal{S}} \langle \phi_n, \gamma s\rangle_{E,E'} \langle \varphi, \gamma s\rangle_{E,E'} \, d\nu(s).$$

In [7, section 3], we show examples where $\mathcal{H}_M$ is of infinite dimension (i.e. $\mathbb{I}_+ = \mathbb{N}$) and where the expression (4.1) can be analytically expressed (i.e. the limit of the sum can be analytically calculated).

Nevertheless, in certain cases, for instance for numerical reasons, we will not be able to consider all the terms in

$$\forall e' \text{ and } f' \in E',\ \langle T_{\mathcal{H}_M} e', f'\rangle_{E,E'} = \sum_{n\in\mathbb{I}_+} \lambda_n \langle \phi_n, e'\rangle_{E,E'} \langle \phi_n, f'\rangle_{E,E'}.$$

We assume that we dispose of an approximated kernel defined from a subset $\mathbb{I}_{app}$ of $\mathbb{I}_+$, i.e.

$$\forall e' \text{ and } f' \in E',\ \left\langle T_{\mathcal{H}_M^{app}} e', f'\right\rangle_{E,E'} = \sum_{n\in\mathbb{I}_+} \lambda_n \langle \phi_n, e'\rangle_{E,E'} \langle \phi_n, f'\rangle_{E,E'}.$$

Hence, for $\varphi \in \mathcal{H}$, we obtain an approximation of the optimal interpolator $P_{\mathcal{H}_M}[\varphi]$ that we note $P_{\mathcal{H}_M^{app}}[\varphi]$:

$$\begin{aligned} \forall e' \in E',\ \left\langle P_{\mathcal{H}_M^{app}}[\varphi], e'\right\rangle_{E,E'} &= \left(\varphi \middle| T_{\mathcal{H}_M^{app}} e'\right)_{\mathcal{H}} \\ &= \sum_{n\in\mathbb{I}_{app}} \sqrt{\lambda_n}\phi_n(x) \left(\sqrt{\lambda_n}\phi_n \middle| \varphi\right)_{\mathcal{H}}, \end{aligned}$$

with $\mathcal{H}_M^{app}$ the closure in $\mathcal{H}$ of the subspace spanned by the ϕ_n, $n \in \mathbb{I}_{app}$.

We pose $\mathbb{I}_{err} = \mathbb{I}_+ \backslash \mathbb{I}_{app}$. Hence, we find the following expression for the approximation error, for all $e' \in E'$,

$$\begin{aligned} \left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{app}}[\varphi], e'\right\rangle_{E,E'} &= \left(\varphi | T_{\mathcal{H}_M} e' - T_{\mathcal{H}_M^{app}} e'\right)_{\mathcal{H}} \\ &= \left(\varphi \middle| \sum_{n\in\mathbb{I}_{err}} \lambda_n \langle \phi_n, e'\rangle_{E,E'} \phi_n\right)_{\mathcal{H}}. \end{aligned} \tag{4.2}$$

Finally, applying the Cauchy-Schwarz inequality to equation (4.2) allows us to control the error of approximation :

Theorem 4.2

$$\forall e' \in E',\ \left|\left\langle \mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{app}}[\varphi], e' \right\rangle_{E,E'}\right|^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}} \lambda_n \langle \phi_n, e' \rangle_{E,E'}^2 \tag{4.3}$$

$$\text{and } \left\|\mathbb{P}_{\mathcal{H}_M}[\varphi] - \mathbb{P}_{\mathcal{H}_M^{app}}[\varphi]\right\|_{\gamma,\nu}^2 \leqslant \|\varphi\|_{\mathcal{H}}^2 \sum_{n \in \mathbb{I}_{err}} \lambda_n. \tag{4.4}$$

We call $\sum_{n \in \mathbb{I}_{err}} \lambda_n$ the **spectral error term**.

Remark 4.1 : In the most part of application cases, the analytical spectral decomposition of $\mathcal{L}_\nu$ would be unknown. Nevertheless, one should dispose of a numerical approximation of this spectral decomposition, obtained from an algorithm of spectral approximation. It should be interesting to study the behavior of theorems 4.1 and 4.2 in such cases. ■

5. Finite Case

We suppose that M is of finite dimension, i.e. $M = \{\mu_1, \cdots, \mu_n\}$, $n \in \mathbb{N}$. We also assume, for simplicity and without loss of generality, that the $\mu_i \in E'$ are such that the symmetric and positive matrix

$$\mathbf{T} \in \mathbb{R}^{n \times n},\ \text{defined by, for } 1 \leqslant i,j \leqslant n,\ \mathbf{T}_{i,j} = (T_{\mathcal{H}}\mu_i | T_{\mathcal{H}}\mu_j)_{\mathcal{H}} \text{ is invertible.} \tag{5.1}$$

For convenience, we introduce the matricial type notations

$$\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)^T \text{ and } \mathbf{T} = \left(T_{\mathcal{H}}\boldsymbol{\mu} | T_{\mathcal{H}}\boldsymbol{\mu}^T\right)_{\mathcal{H}} = \left\langle T_{\mathcal{H}}\boldsymbol{\mu}, \boldsymbol{\mu}^T \right\rangle = \left\langle \boldsymbol{\mu}, T_{\mathcal{H}}\boldsymbol{\mu}^T \right\rangle,$$

where $T_{\mathcal{H}}\boldsymbol{\mu} = (T_{\mathcal{H}}\mu_1, \cdots, T_{\mathcal{H}}\mu_n)^T$. Hence, for $\varphi \in \mathcal{H}$, the optimal interpolator of φ for M can be written under the form :

$$h_{\varphi,M} = T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{T}^{-1} \langle \boldsymbol{\mu}, \varphi \rangle, \tag{5.2}$$

with $\langle \boldsymbol{\mu}, \varphi \rangle = \left(\langle \varphi, \mu_1 \rangle_{E,E'}, \cdots, \langle \varphi, \mu_n \rangle_{E,E'}\right)^T$. Remark for instance that, with our notation,

$$\langle \boldsymbol{\mu}, \varphi \rangle^T = \left\langle \varphi, \boldsymbol{\mu}^T \right\rangle, \text{and for } e \in E \text{ and } e' \in E',\ \langle e, e' \rangle_{E,E'} = \langle e, e' \rangle = \langle e', e \rangle,$$

so, we will write, for $e' \in E'$, $\langle e', h_{\varphi,M} \rangle_{E,E'} = \left\langle e', T_{\mathcal{H}}\boldsymbol{\mu}^T \right\rangle \mathbf{T}^{-1} \langle \boldsymbol{\mu}, \varphi \rangle$.

The aim of this section is to proof, by direct calculation, that the expression of the optimal interpolator given in equation (5.2) is equal to the one given in theorem 4.1.

The linear operator, of remark 3.6 type, L_ν is defined by

$$\forall h \in \mathcal{H},\ L_\nu[h] = \sum_{i=1}^{n} w_i \langle h, \mu_i \rangle_{E,E'} T_{\mathcal{H}}\mu_i, \tag{5.3}$$

with $w_i > 0$ for all $1 \leqslant i \leqslant n$. Introducing the matrix $\mathbf{W} = diag\,(w_1, \cdot, w_n)$, L_ν can be written under the matricial form :

$$\forall h \in \mathcal{H},\ L_\nu[h] = T_{\mathcal{H}} \boldsymbol{\mu}^T \mathbf{W} \langle \boldsymbol{\mu}, h \rangle .$$

The finiteness of the sum (5.3) assures us that L_ν verifies the whole properties required in section 3. The symmetric and positive bilinear form $(\cdot|\cdot)_{\gamma,\nu}$ on $\mathcal{H}$ and associated with L_ν by equation (3.11) is, for h and $g \in \mathcal{H}$,

$$(h|g)_{\gamma,\nu} = \sum_{i=1}^{n} w_i \langle h, \mu_i \rangle_{E,E'} \langle g, \mu_i \rangle_{E,E'} = \langle h, \boldsymbol{\mu}^T \rangle \mathbf{W} \langle \boldsymbol{\mu}, g \rangle .$$

Remark 5.1 : We easily re obtain the properties of self-adjunction and symmetry of L_ν by matricial calculations :

$$\begin{aligned} \forall h \text{ and } k \in \mathcal{H},\ (h|L_\nu[k])_{\mathcal{H}} &= \left(h|T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W} \langle \boldsymbol{\mu}, k \rangle\right)_{\mathcal{H}} = \left(h|T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W} \left(T_{\mathcal{H}}\boldsymbol{\mu}, k\right)_{\mathcal{H}}\right)_{\mathcal{H}} \\ &= \left(h|T_{\mathcal{H}}\boldsymbol{\mu}^T\right)_{\mathcal{H}} \mathbf{W} \left(T_{\mathcal{H}}\boldsymbol{\mu}, k\right)_{\mathcal{H}} = \left(L_\nu[h]|k\right)_{\mathcal{H}} , \end{aligned}$$

$$\begin{aligned} \forall h \text{ and } k \in \mathcal{H},\ (h|L_\nu[k])_{\gamma,\nu} &= \left(h|T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W} \langle \boldsymbol{\mu}, k \rangle\right)_{\gamma,\nu} = \langle h, \boldsymbol{\mu}^T \rangle \mathbf{W} \left\langle \boldsymbol{\mu}, T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W} \langle \boldsymbol{\mu}, k \rangle \right\rangle \\ &= \langle h, \boldsymbol{\mu}^T \rangle \mathbf{W} \left\langle T_{\mathcal{H}}\boldsymbol{\mu}, \boldsymbol{\mu}^T \right\rangle \mathbf{W} \langle \boldsymbol{\mu}, k \rangle = \left(L_\nu[h]|k\right)_{\gamma,\nu} , \end{aligned}$$

$$\forall h \text{ and } k \in \mathcal{H},\ (h|L_\nu[k])_{\mathcal{H}} = \left(h|T_{\mathcal{H}}\boldsymbol{\mu}^T\right) \mathbf{W} \langle \boldsymbol{\mu}, k \rangle = \langle h, \boldsymbol{\mu}^T \rangle \mathbf{W} \langle \boldsymbol{\mu}, k \rangle = (h|k)_{L^2} . \quad \blacksquare$$

For $\boldsymbol{\alpha} \in \mathbb{R}^n$, let $\widehat{\phi}_{\boldsymbol{\alpha}} = T_{\mathcal{H}}\boldsymbol{\mu}^T \boldsymbol{\alpha} \in \mathcal{H}_M$. We have

$$L_\nu\left[\widehat{\phi}_{\boldsymbol{\alpha}}\right] = T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W} \left\langle \boldsymbol{\mu}, T_{\mathcal{H}}\boldsymbol{\mu}^T \right\rangle \boldsymbol{\alpha} = T_{\mathcal{H}}\boldsymbol{\mu}^T \mathbf{W}\mathbf{T}\boldsymbol{\alpha}.$$

Hence, the eigenvalues of L_ν on $\mathcal{H}_M$ and the ones of $\mathbf{WT}$ are the same. Let $\lambda_1, \cdots, \lambda_n > 0$ be those eigenvalues (the strict positivity is a consequence of the hypothesis (5.1)) and let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ their associated eigenvectors, i.e. $\mathbf{WT} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$ with $\boldsymbol{\Lambda} = diag\,(\lambda_1, \cdot, \lambda_n)$ and $\mathbf{P} = (\mathbf{v}_1|\cdots|\mathbf{v}_n)$. We have, for all $1 \leqslant i \leqslant n$,

$$L_\nu\left[\widehat{\phi}_{\mathbf{v}_i}\right] = \lambda_i \widehat{\phi}_{\mathbf{v}_i}.$$

We define :

$$\widehat{\boldsymbol{\phi}} = \left(\widehat{\phi}_{\mathbf{v}_1}, \cdots, \widehat{\phi}_{\mathbf{v}_n}\right)^T = \mathbf{P}^T T_{\mathcal{H}}\boldsymbol{\mu}.$$

We have :

$$\left(\widehat{\boldsymbol{\phi}}|\widehat{\boldsymbol{\phi}}^T\right)_{\mathcal{H}} = \mathbf{P}^T\mathbf{T}\mathbf{P} \text{ and } \left(\widehat{\boldsymbol{\phi}}|\widehat{\boldsymbol{\phi}}^T\right)_{\gamma,\nu} = \mathbf{P}^T\mathbf{TWTP} = \mathbf{P}^T\mathbf{TP}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\mathbf{P}^T\mathbf{TP}. \tag{5.4}$$

From equation (5.4), the matrix $\mathbf{P}^T\mathbf{T}\mathbf{P}$ is diagonal, which is then also the case for $\mathbf{P}^T\mathbf{T}\mathbf{W}\mathbf{T}\mathbf{P}$. As expected, the relation $\left\|\widehat{\phi}_{\mathbf{v}_i}\right\|_{\gamma,\nu} = \lambda_i \left\|\widehat{\phi}_{\mathbf{v}_i}\right\|_{\mathcal{H}}$ is verified.

Thus, the eigenfunctions (normalized) of L_ν on $\mathcal{H}_M$, $(\cdot|\cdot)_{\gamma,\nu}$ are the components of the vector

$$\boldsymbol{\phi} = \left(\mathbf{P}^T\mathbf{T}\mathbf{W}\mathbf{T}\mathbf{P}\right)^{-\frac{1}{2}} \widehat{\boldsymbol{\phi}}.$$

Finally, we obtain

$$\begin{aligned}
T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{T}^{-1}\langle\boldsymbol{\mu},\varphi\rangle &= T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{T}^{-1}\mathbf{W}^{-1}\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle = T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}^{-1}\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle \\
&= T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{P}\boldsymbol{\Lambda}^{-1}\mathbf{P}^{-1}\mathbf{T}^{-1}\mathbf{P}^{-T}\mathbf{P}^T\mathbf{T}\mathbf{M}\langle\boldsymbol{\mu},\varphi\rangle \\
&= T_{\mathcal{H}}\boldsymbol{\mu}^T\mathbf{P}\left(\mathbf{P}^T\mathbf{T}\mathbf{W}\mathbf{T}\mathbf{P}\right)^{-1}\mathbf{P}^T\mathbf{T}\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle \\
&= \widehat{\boldsymbol{\phi}}^T\left(\mathbf{P}^T\mathbf{T}\mathbf{W}\mathbf{T}\mathbf{P}\right)^{-1}\left\langle\widehat{\boldsymbol{\phi}},\boldsymbol{\mu}^T\right\rangle\mathbf{W}\langle\boldsymbol{\mu},\varphi\rangle \\
&= \sum_{k=1}^{n}\phi_k\int_{\mathcal{S}}\langle\phi_k,\gamma s\rangle_{E,E'}\langle\varphi,\gamma s\rangle_{E,E'}\,d\nu(s).
\end{aligned}$$

6. Example of application

Let $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{H}$ be the RKHS of real-valued functions on $\mathcal{X}$ (see remark 3.7) associated with the kernel (squared exponential or Gaussian kernel, see for example [3]), for x and $y \in \mathcal{X}$,

$$K(x,y) = e^{-\frac{\|x-y\|^2}{\sigma^2}}, \quad \text{with } \sigma > 0 \text{ and } \|\cdot\| \text{ the euclidean norm.}$$

For $m \in \mathbb{N}$, we consider $\mathcal{E}^m \subset \mathbb{R}^{\mathcal{X}}$ the subspace of functions of class C^m endowed with the topology of the uniform convergence on the compact subsets of $\mathcal{X}$ for all the derivatives of order $\leqslant m$ (or of general order if $m = +\infty$). Then, from [1, proposition 25], for all $m \in \mathbb{N}$ (and also for $m = +\infty$), $\mathcal{H}$ is a Hilbert subspace of $\mathcal{E}^m$.

In, [5], we have considered the case where $M = \mathit{span}\{\delta_s,\ s \in \mathcal{S}\}$, with $\mathcal{S} \subset \mathbb{R}^2$ the circle of center 0 and radius $R > 0$ (*Dirichlet condition* on the circle $\mathcal{S}$). In this present work, our aim is, in a first time, to impose constraints to the normal derivative of the interpolator on the circle (*Neumann condition*), which is possible because $\mathcal{H}$ in a Hilbert subspace of $\mathcal{E}^1$. Following remark 2.1, we next combine our results with the ones of [5] to obtain a model in which both value and normal derivative are controlled on the circle (*Robin condition*).

6.1. *Derivative Constraint*

We consider $L^2(\mathcal{S},\nu) = L^2([0,2\pi])$, the space of 2π-periodic squared integrable functions (with respect to the Lebesgue measure) on $[0,2\pi]$, endowed with the Hilbert norm

$$\forall f \in L^2([0,2\pi]),\ \|f\|_{L^2}^2 = \int_0^{2\pi} f(\theta)^2 R d\theta.$$

Let $x = (x_1,x_2)$ be a point of $\mathcal{X}$, for convenience, we will use a polar coordinates system, that is $x = (r_x\cos\alpha_x, r_x\sin\alpha_x)$ with $r_x \in \mathbb{R}_+$ and $\alpha_x \in [0,2\pi]$. For $x \in \mathcal{X}$, let us define $\mu_x \in E'$ such that, for all $h \in \mathcal{H}$,

$$\langle h, \mu_x \rangle_{E,E'} = \frac{\partial}{\partial r_x} h(x).$$

We consider $M = span\{\mu_s,\ s \in \mathcal{S}\}$. Then, $\gamma : \mathcal{S} \to M \subset E'$ is given by $\gamma s = \mu_s$ and compactness and continuity arguments assure us that $\mathcal{H}_M$ is separable.

The polar coordinates expression of the kernel $K(\cdot, \cdot)$ is

$$K(x, y) = e^{-\frac{1}{\sigma^2}\left(r_x^2 + r_y^2 - 2r_x r_y \cos(\alpha_x - \alpha_y)\right)}, \text{ then}$$

$$\frac{\partial}{\partial r_x} K(x, y) = -\frac{2}{\sigma^2}\big(r_x - r_y \cos(\alpha_x - \alpha_y)\big)K(x, y) \text{ and}$$

$$\begin{aligned}\frac{\partial^2}{\partial r_y \partial r_x} K(x, y) = {} & \frac{2}{\sigma^2}\cos(\alpha_x - \alpha_y)K(x, y) \\ & + \frac{4}{\sigma^4}\big(r_x - r_y \cos(\alpha_x - \alpha_y)\big)\big(r_y - r_x \cos(\alpha_x - \alpha_y)\big)K(x, y).\end{aligned}$$

For $f \in L^2([0, 2\pi])$, our "problem adapted" integral operator is, for $\alpha \in [0, 2\pi]$,

$$\mathcal{L}_\nu[f](\alpha) = \int_0^{2\pi} \frac{\partial^2}{\partial r_s \partial r_x} K\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta, \tag{6.1}$$

with $s_{R,\theta} = (R\cos\theta, R\sin\theta)$ and $x_{R,\alpha} = (R\cos\alpha, R\sin\alpha)$.

Let us pose $A = \frac{2}{\sigma^2} - \frac{8R^2}{\sigma^4}$ and $B = \frac{4R^2}{\sigma^4}$. Then, straightforward calculations show (using parity arguments) that the eigenvalues of $\mathcal{L}_\nu$ are

$$\text{for } n \geqslant 0,\ \lambda_n = \int_0^{2\pi} \left[A\cos\theta + B(1 + \cos^2\theta)\right] e^{-\frac{2R^2}{\sigma^2}(1-\cos\theta)} \cos(n\theta) R d\theta.$$

For $n = 0$, λ_0 is associated with the eigenfunction, for $\alpha \in [0, 2\pi]$, $\widetilde{\phi}_0(\alpha) = \frac{1}{\sqrt{2\pi R}}$.

For $n \geqslant 1$, λ_n are of multiplicity 2, associated with, for $\alpha \in [0, 2\pi]$,

$$\widetilde{\phi}_n^c(\alpha) = \frac{1}{\sqrt{\pi R}} \cos n\alpha, \text{ and } \widetilde{\phi}_n^s(\alpha) = \frac{1}{\sqrt{\pi R}} \sin n\alpha.$$

Finally, the spectral decomposition of $\mathcal{L}_\nu$ is completed because the eigenfunctions $\widetilde{\phi}_0$, $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$ is a Hilbert basis of $L^2([0, 2\pi])$.

We next define the regularized eigenfunctions ϕ_0, ϕ_n^c and $\phi_n^s \in \mathcal{H}_M$ (see equation (3.10)). In our case with have, for instance,

$$\forall n \geqslant 1,\ \forall x \in \mathcal{X},\ \phi_n^c(x) = \frac{1}{\lambda_n} \int_0^{2\pi} \frac{\partial}{\partial r_s} K(s_{R,\theta}, x) \frac{\cos(n\theta)}{\sqrt{\pi R}} R d\theta. \tag{6.2}$$

Remark 6.1 : Let us consider the values of the eigenfunctions ϕ_0, ϕ_n^c and ϕ_n^s on the circle $\mathcal{S}$. From equation (6.2), it appears that those ones are linked with the integral operator, for $x_{R,\alpha} = (R\cos\alpha, R\sin\alpha)$ and $f \in L^2([0, 2\pi])$,

$$J_\nu[f](\alpha) = \int_0^{2\pi} \frac{\partial}{\partial r_s} K\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta. \tag{6.3}$$

J_ν is not self-adjoint and positive, but we remark that for, $n \geqslant 0$,

$$\rho_n = \int_0^{2\pi} \frac{-2R}{\sigma^2}(1-\cos\theta)e^{-\frac{2R^2}{\sigma^2}(1-\cos\theta)}\cos(n\theta)Rd\theta, \quad (\text{with } \rho_n \in \mathbb{R})$$

are eigenvalues of J_ν. For $n = 0$, ρ_0 is associated with the same eigenfunction than λ_0, that is for $\alpha \in [0, 2\pi]$, $\widetilde{\phi}_0(\alpha) = \frac{1}{\sqrt{2\pi R}}$.

For $n \geqslant 1$, the ρ_n are also associated with the same eigenfunctions than λ_n, so the ρ_n are of multiplicity 2, associated with, for $\alpha \in [0, 2\pi]$,

$$\widetilde{\phi}_n^c(\alpha) = \frac{1}{\sqrt{\pi R}}\cos n\alpha, \text{ and } \widetilde{\phi}_n^s(\alpha) = \frac{1}{\sqrt{\pi R}}\sin n\alpha.$$

The eigenfunctions $\widetilde{\phi}_n$ of J_ν form an Hilbert basis of $L^2([0, 2\pi])$, hence, the operator J_ν is diagonalizable on $L^2([0, 2\pi])$, its spectrum is composed by the eigenvalues ρ_n associated with $\widetilde{\phi}_0$, $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$.

We finally obtain that the values of the regularized eigenfunctions of $\mathcal{L}_\nu$ on the circle $\mathcal{S}$ are, for $x_{R,\alpha} = (R\cos\alpha, R\sin\alpha)$,

$$\phi_0(x_{R,\alpha}) = \frac{\rho_0}{\lambda_0}\widetilde{\phi}_0(\alpha),\ \phi_n^c(x_{R,\alpha}) = \frac{\rho_n}{\lambda_n}\widetilde{\phi}_n^c(\alpha) \text{ and } \phi_n^s(x_{R,\alpha}) = \frac{\rho_n}{\lambda_n}\widetilde{\phi}_n^s(\alpha).$$

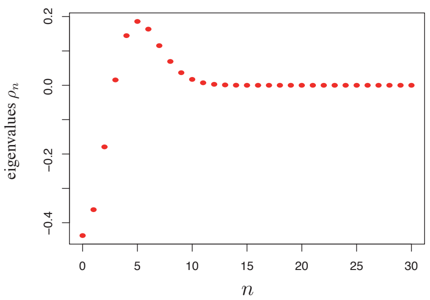

Figure 2: Graphical representation of ρ_n, $0 \leqslant n \leqslant 30$ for $R = 3$ and $\sigma^2 = 2$.

■

Numerical Applications. We fix $R = 3$ and $\sigma^2 = 2$, we next compute the eigenvalues and regularized eigenfunctions of $\mathcal{L}_\nu$. The obtained results are represented on figures 3 and 4, integrals have been performed with Monte-Carlo algorithms.

Following section 4, we approximate the kernel $T_{\mathcal{H}_M}$ with the 31 eigenfunctions associated with the most important eigenvalues, that is, for all x and $t \in \mathbb{R}^2$

$$K_M^{app}(x,t) = \lambda_0\phi_0(x)\phi_0(t) + \sum_{n=1}^{15}\lambda_n\left[\phi_n^c(x)\phi_n^c(t) + \phi_n^s(x)\phi_n^s(t)\right],$$

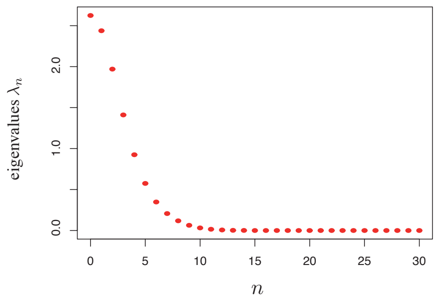

Figure 3: Graphical representation of $\mathcal{L}_\nu$ spectrum, $0 \leqslant n \leqslant 30$, $R = 3$ and $\sigma^2 = 2$.

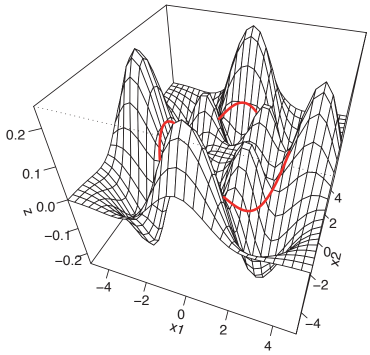

Figure 4: Eigenfunction ϕ_n^c on $[-5, 5]^2$ for n =4.

with $K_M^{app}(x,t) = \left\langle T_{\mathcal{H}_M^{app}} \delta_x, \delta_t \right\rangle_{E,E'}$. We have

$$Trace\,(\mathcal{L}_\nu) = \frac{4\pi R}{\sigma^2} = \texttt{18.84956} \text{ and } \sum_{k \in \mathbb{I}_{app}} \lambda_k = \lambda_0 + 2 \sum_{n=1}^{15} \lambda_n = \texttt{18.84928}.$$

Hence, we obtain for the spectral error term :

$$\sum_{k \in \mathbb{I}_{err}} \lambda_k = \texttt{2.797111e-04}. \tag{6.4}$$

We next approximate the kernel $K_0(\cdot,\cdot) = K(\cdot,\cdot) - K_M(\cdot,\cdot)$ of the sub-RKHS $\mathcal{H}_0$ with, for all x and $t \in \mathbb{R}^2$,

$$K_0^{app}(x,t) = K(x,t) - K_M^{app}(x,t).$$

We remind that $\mathcal{H}_0$ is the subspace of functions $h \in \mathcal{H}$ such that $\frac{\partial}{\partial r_s} h(s) = 0$ for all $s \in \mathcal{S}$.

Figure 5 shows the sample path of a centered Gaussian process with covariance $K_0^{app}(\cdot,\cdot)$. As expected, this approximates a centered Gaussian process of covariance $K(\cdot,\cdot)$ conditioned to have null normal derivative on $\mathcal{S}$. In this example, instead of being null, each normal derivative on $\mathcal{S}$ follows a centered normal distribution with variance

$$\frac{1}{2\pi R} \sum_{k \in \mathbb{I}_{err}} \lambda_k = \texttt{1.483913e-05}.$$

Remark 6.2 : For $s \in \mathcal{S}$, we have

$$K_0^{app}(s,s) = 1 - \frac{\rho_0^2}{\lambda_0 2\pi R} - \sum_{n=1}^{15} \frac{\rho_n^2}{\lambda_n \pi R} = \texttt{0.9568043}.$$

■

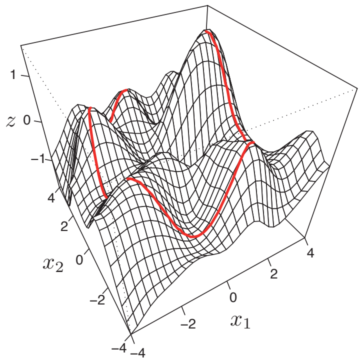

Figure 5: Sample path of a centered Gaussian process with covariance $K_0^{app}(\cdot,\cdot)$.

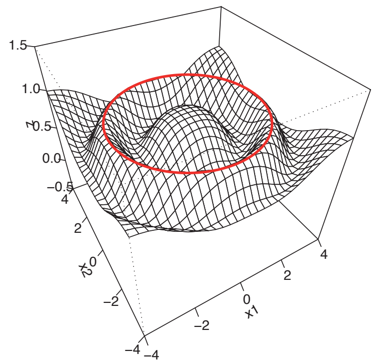

Figure 6: Graphical representation of $x \mapsto K_0^{app}(x,x)$ on $[-4,4]^2$.

6.2. Double Constraint

In this last section, the results of the preceding section and the ones of [5] are combined in order to obtain a model that takes into account the values of the function and of its normal derivative on $\mathcal{S}$. We proceed in a same way than in remark 2.1. We first consider the problem of the normal derivatives and, in a second time, the problem of the values. We present an efficient way to approximate the kernel $K_{00}(\cdot,\cdot)$ of the subspace $\mathcal{H}_{00}$ of functions $h \in \mathcal{H}$ such that

$$\forall s \in \mathcal{S},\ \langle h, \mu_s\rangle_{E,E'} = 0 \text{ and } \langle h, \delta_s\rangle_{E,E'} = 0.$$

Nevertheless, in what follows, one can find the necessary informations allowing to treat the general problem (that is imposed constraints on the values and on the normal derivatives).

We rename $\mathcal{L}_\nu^N$ the operator of equation (6.1) (N for *Neumann*). We also pose, for $n \geqslant 1$, $\lambda_n^N = \lambda_n$.

In [5], we have study the integral operator $\mathcal{L}_\nu^D$ (D for *Dirichlet*) on $L^2([0,2\pi])$ associated to $K(\cdot,\cdot)$,

$$\mathcal{L}_\nu^D[f](\alpha) = \int_0^{2\pi} K\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta.$$

From [5], we know that the eigenfunctions of $\mathcal{L}_\nu^D$ are the same than the one $\mathcal{L}_\nu^N$ ($\mathcal{L}_\nu^N$ and $\mathcal{L}_\nu^D$ are diagonalizable in the same basis, which is also the case for the operator J_ν of remark 6.1). For $n \in \mathbb{N}$, let us denote λ_n^D the eigenvalues of $\mathcal{L}_\nu^D$.

We consider the integral operator on $L^2([0,2\pi])$ associated to the kernel $K_0(\cdot,\cdot)$ of section 6.1, that is,

$$\mathcal{L}_\nu^R[f](\alpha) = \int_0^{2\pi} K_0\left(x_{R,\alpha}, s_{R,\theta}\right) f(\theta) R d\theta,$$

(R for *Robin*). From remark 6.1, we find that the eigenvalues λ_n^R, $n \in \mathbb{N}$, of $\mathcal{L}_\nu^R$ are given by

$$\lambda_n^R = \lambda_n^D - \frac{\rho_n^2}{\lambda_n^N}.$$

λ_0^R is associated with $\widetilde{\phi}_0$, for $n \geqslant 1$, λ_n^R is of multiplicity 2 and associated with $\widetilde{\phi}_n^c$ and $\widetilde{\phi}_n^s$.

Considering the operator $\mathcal{L}_\nu^N$, we respectively denote ϕ_0^N, ϕ_n^{cN} and ϕ_n^{sN} its regularized eigenfunctions. For example, we have, for $n \geqslant 1$,

$$\phi_n^{cN} = \frac{1}{\lambda_n^N} \mathcal{L}_\nu^N [\widetilde{\phi}_n^c].$$

In a same way, we denote ϕ_0^D, ϕ_n^{cD} and ϕ_n^{sD} the regularized eigenfunctions of $\mathcal{L}_\nu^D$. We finally introduce ϕ_0^R, ϕ_n^{cR} and ϕ_n^{sR} the regularized eigenfunctions of $\mathcal{L}_\nu^R$.

Straightforward calculations give that, for all $x \in \mathcal{X}$,

$$\phi_0^R(x) = \frac{1}{\lambda_0^R} \left(\lambda_0^D \phi_0^D(x) - \rho_0 \phi_0^N(x) \right),$$

$$\forall n \geqslant 1, \ \phi_n^{cR}(x) = \frac{1}{\lambda_n^R} \left(\lambda_n^D \phi_n^{cD}(x) - \rho_0 \phi_n^{cN}(x) \right) \text{ and}$$

$$\phi_n^{sR}(x) = \frac{1}{\lambda_n^R} \left(\lambda_n^D \phi_n^{sD}(x) - \rho_0 \phi_n^{sN}(x) \right).$$

As an illustration, we compute the kernel $K_{00}^{app}(\cdot,\cdot)$ given by, for x and $t \in \mathcal{X}$,

$$K_{00}^{app}(x,t) = K_0^{app}(x,t) - \lambda_0^R \phi_0^R(x) \phi_0^R(t) - \sum_{n=1}^{15} \lambda_n^R \left[\phi_n^{cR}(x) \phi_{nR}^c(t) + \phi_n^{sR}(x) \phi_n^{sR}(t) \right].$$

Finally, figure 7 shows the sample path of an centered Gaussian process with covariance $K_{00}^{app}(\cdot,\cdot)$.

As in section 6.1, instead of being null, the normal derivatives on the circle follow a centered normal distribution with variance `1.483913e-05`. For the values of the sample path on $\mathcal{S}$, those ones follow a centered normal distribution with variance, for $s \in \mathcal{S}$,

$$K_{00}^{app}(s,s) = K_0^{app}(s,s) - \frac{1}{2\pi R} \left(\lambda_0^R + 2 \sum_{n=0}^{15} \lambda_n^R \right) = \texttt{1.402309e-06},$$

the value $K_0^{app}(s,s) = \texttt{0.9568043}$ being given in remark 6.2.

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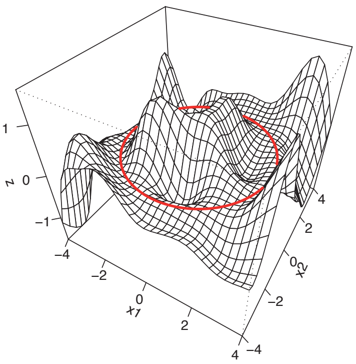

Figure 7: Sample path of a centered Gaussian process with covariance $K_{00}^{app}(\cdot,\cdot)$.

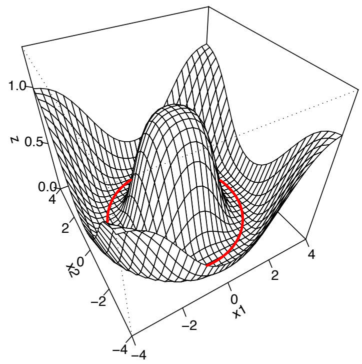

Figure 8: Graphical representation of $x \mapsto K_{00}^{app}(x,x)$ on $[-4,4]^2$.

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