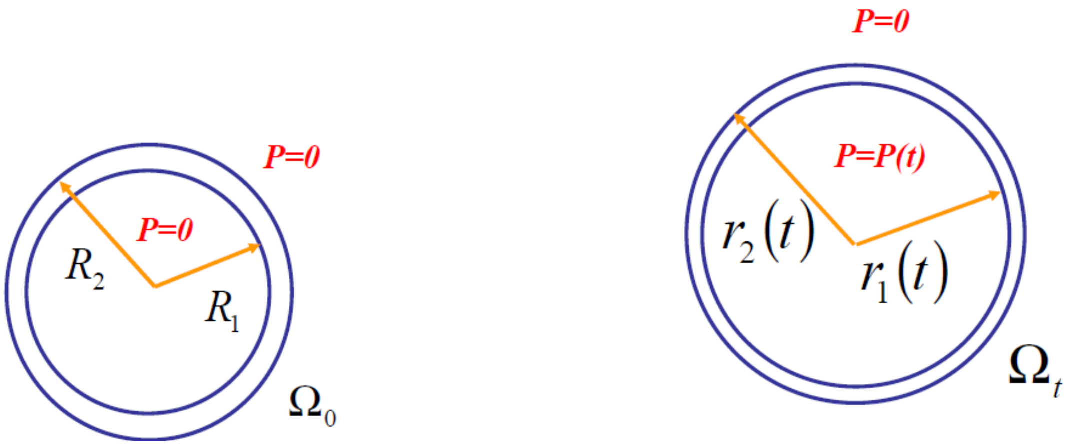
Inflation of a hyperelastic spherical balloon



Topics in finite elasticity: Hyperelasticity of rubber, elastomers, and biological tissues—with examples

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11. INFLATION RESPONSE OF A BALLOON

Problem is solved in spherical coordinate system

Local orthonormal frame (basis vectors) is denoted $(E_r, E_\theta, E_\varphi)$ in the undeformed reference configuration and $(e_r, e_\theta, e_\varphi)$ in the deformed configuration

No shear => $(E_r, E_\theta, E_\varphi) = (e_r, e_\theta, e_\varphi)$

$$F(r) = \begin{bmatrix} \frac{\partial r}{\partial R} & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & \frac{r}{R} \end{bmatrix}$$

$$\frac{H}{R_1} = \gamma$$

Thin balloon: $\gamma \ll 1$

$$\lambda^2 \mu = 1$$

$$F(r) = F = \begin{bmatrix} \frac{\partial r}{\partial R} \approx \mu & 0 & 0 \\ 0 & \frac{r}{R} \approx \lambda & 0 \\ 0 & 0 & \frac{r}{R} \approx \lambda \end{bmatrix}$$

$$F = \begin{bmatrix} 1/\lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Cauchy stress:

$$\sigma = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\varphi\varphi} \end{bmatrix}$$

$$-p \leq \sigma_{rr} \leq 0$$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = ?$$

Laplace law:

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{Pr_1}{2h} = \frac{P\lambda^3 R_1}{2H} = \frac{P\lambda^3}{2\gamma}$$

$$\sigma = P\begin{bmatrix} <1 & 0 & 0 \\ 0 & \frac{\lambda^3}{2\gamma} & 0 \\ 0 & 0 & \frac{\lambda^3}{2\gamma} \end{bmatrix}$$

Thin balloon

$$\gamma \ll 1$$

$$\frac{1}{\gamma} \gg 1$$

$$\sigma_{rr} \ll \sigma_{\theta\theta}$$

$$\sigma_{rr} \ll \sigma_{\varphi\varphi}$$

$$\sigma \approx \frac{P\lambda^3 R_1}{2H}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For Neo Hooke

$\psi(I_1 - 3) = \frac{\mu_0}{2}(I_1 - 3)$

Equilibrium

$$\sigma = \begin{bmatrix} \frac{\mu_0}{\lambda^4} + c & 0 & 0 \\ 0 & \mu_0\lambda^2 + c & 0 \\ 0 & 0 & \mu_0\lambda^2 + c \end{bmatrix} = \frac{P\lambda^3 R_1}{2H}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_{rr} \approx 0$$

$$c = -\frac{\mu_0}{\lambda^4}$$

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_0\lambda^2 - \frac{\mu_0}{\lambda^4} & 0 \\ 0 & 0 & \mu_0\lambda^2 - \frac{\mu_0}{\lambda^4} \end{bmatrix} = \mu_0\left(\lambda^2 - \frac{1}{\lambda^4}\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally:

$$\frac{P\lambda^3 R_1}{2H} = \mu_0 \left(\lambda^2 - \frac{1}{\lambda^4}\right)$$

$$P = \mu_0 \frac{2H}{R_1}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right)$$

$$\lambda^* = \frac{r^*}{r_0} = \sqrt[6]{7} = 1.383.$$

Strain energy density of biological tissue

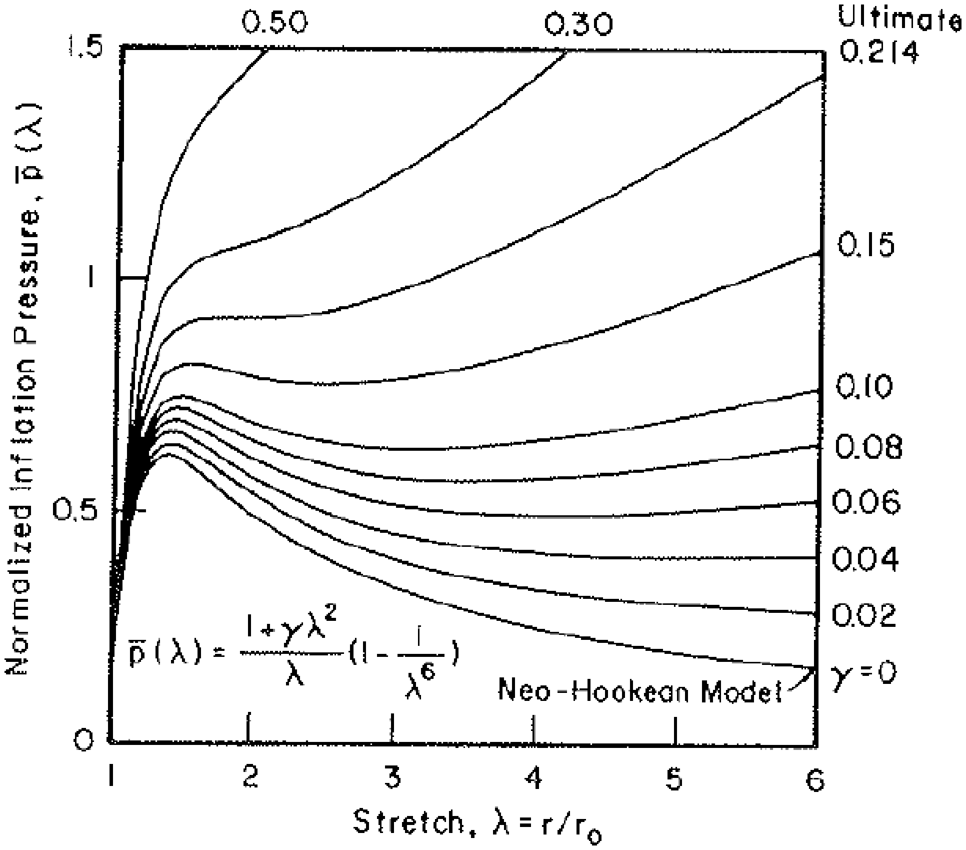
$$\psi(I_1 - 3) = \frac{\mu_0}{2\gamma}\left[e^{\gamma(I_1-3)} - 1\right]$$

$$\sigma = \mu_0 e^{\gamma\left(2\lambda^2 + \frac{1}{\lambda^4} - 3\right)} \left(\lambda^2 - \frac{1}{\lambda^4}\right)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \mu_0 e^{\gamma\left(2\lambda^2 + \frac{1}{\lambda^4} - 3\right)} \frac{2H}{R_1}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right)$$

When λ becomes large

$$P \approx \mu_0 e^{\gamma(2\lambda^2 - 3)} \frac{2H}{\lambda R_1}$$



0.50
0.30
Ultimate
0.214
1.5
1
0.5
0
Normalized Inflation Pressure, $\bar{p}(\lambda)$
0.15
0.10
0.08
0.06
0.04
0.02
$\gamma = 0$
Neo-Hookean Model
$\bar{p}(\lambda) = \frac{1+\gamma\lambda^2}{\lambda}(1 - \frac{1}{\lambda^6})$
1
2
3
4
5
6
Stretch, $\lambda = r/r_0$