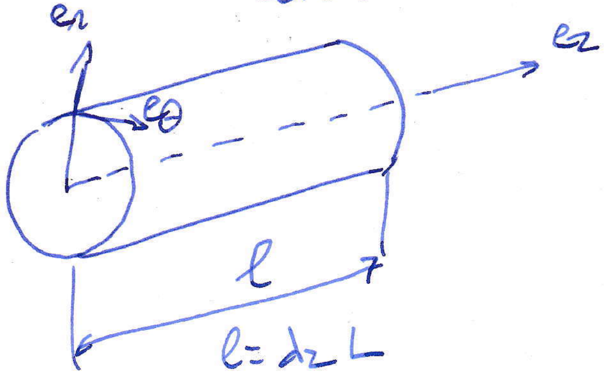
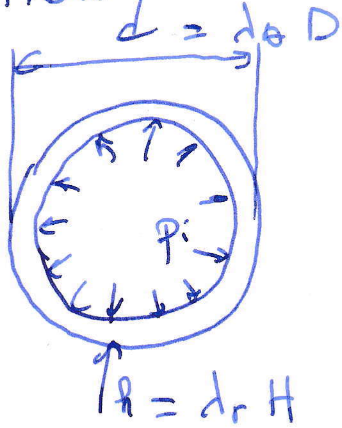
- CURRENT CONFIGURATION -

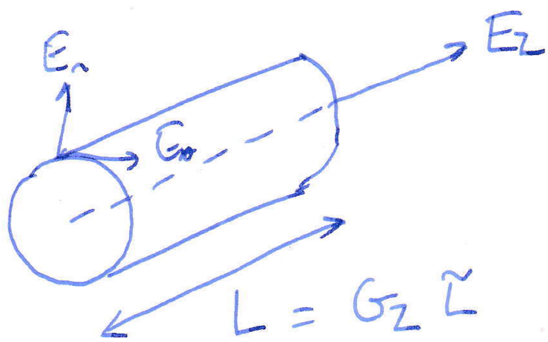


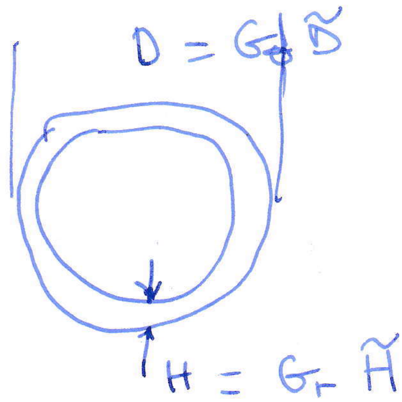


deformation gradient F (non growth related)

Laplace law : $\sigma_{\theta\theta} = \dfrac{p_i\, d}{2h}$

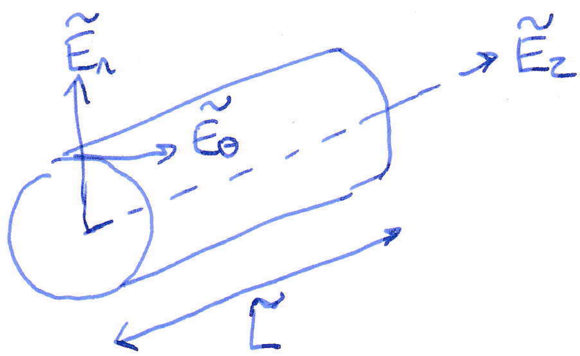
- INTERMEDIATE CONFIGURATION

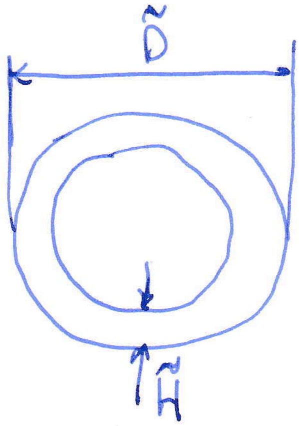




deformation gradient G

- REFERENCE CONFIGURATION





a) G is the deformation gradient related to growth (from the reference to the intermediate configuration)

$$G = \begin{bmatrix} G_r & 0 & 0 \\ 0 & G_\theta & 0 \\ 0 & 0 & G_z \end{bmatrix}$$

$$= \underbrace{G_r \, E_r \otimes \tilde{E_r}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + \underbrace{G_\theta \, E_\theta \otimes \tilde{E_\theta}}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} + \underbrace{G_z \, E_z \otimes \tilde{E_z}}_{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Rk. The constant volume growth would be:

$$J = \det(G) = \text{constante}$$

$$\frac{\partial}{\partial t}\left(\det(G)\right) = 0$$

$$Tr\left(\mathring{G}\right) = 0$$

Rk The constant density growth is:

$$\rho_0 = \cancel{\rho_0} \text{ constant} = \frac{1}{J} = \frac{1}{\det(G)}$$

$$\frac{\partial \rho_0}{\partial t} = \mathring{\rho_0} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{\det G}\right) = 0$$

$$\Rightarrow Tr\left(\mathring{G}\, G^{-1}\right) = 0$$

In the problem, the rate of volume change is S_v.

This means $\dot{\rho}_0 = Tr(\dot{G} G^{-1}) = S_v$

$$\Rightarrow G_\theta^{-1}\dot{G}_\theta + G_z^{-1}\dot{G}_z + G_r^{-1}\dot{G}_r = S_v$$

or: $$\boxed{\frac{\dot{G}_r}{G_r} + \frac{\dot{G}_\theta}{G_\theta} + \frac{\dot{G}_z}{G_z} = S_v}$$

if isotropic, $G_r = G_\theta = G_z = G$, so $\boxed{S_v = 3\frac{\dot{G}}{G}}$

b) Non growth related deformation gradient:

$$F = \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{bmatrix}$$

$J = 1$

$\lambda_r \lambda_\theta \lambda_z = 1$

$\rightarrow$ no change of volume

$\rightarrow H = \frac{h}{\lambda_r} = \lambda_\theta \lambda_z h$

The question is to relate S_v and the non growth related deformation gradient for homeostatic conditions.

The homeostatic conditions means $\frac{p_i}{2\sigma_\theta}$ is a constant k, such as $h = kd$ (Laplace law)

$$G_r = H/\tilde{H} = \lambda_\theta \lambda_z h/\tilde{H} = \lambda_\theta \lambda_z kd/\tilde{H}$$

isotropic: $G = G_r = \lambda_\theta^2 \lambda_z KD/\tilde{H}$

then $\dot{G} = 2\lambda_\theta \dot{\lambda}_\theta \lambda_z KD/\tilde{H}$ $\rightarrow$ *

Finally:

$$\frac{\dot{G}}{G} = \frac{2\, \cancel{\lambda_\theta}\, \dot{\lambda}_\theta\, \cancel{\lambda_z}\, \cancel{K}\cancel{D} / \cancel{H}}{\lambda_\theta^2\, \cancel{\lambda_z}\, \cancel{K}\cancel{D} / \cancel{H}}$$

$$\frac{\dot{G}}{G} = \frac{2\, \dot{\lambda}_\theta}{\lambda_\theta}$$

$$\boxed{S_v = 3\,\frac{\dot{G}}{G} = 6\,\frac{\dot{\lambda}_\theta}{\lambda_\theta}}$$

* Note that the derivative of λ_z with time $\frac{\partial \lambda_z}{\partial t} = \dot{\lambda}_z$ is zero ($\dot{\lambda}_z = 0$) as $\lambda_z = \ell / L$ is a constant.

CONCLUSION.

For any elastic stretch variation $\dot{\lambda}_\theta$, the homeostasis is maintained if:

$$S_v = 6\,\frac{\dot{\lambda}_\theta}{\lambda_\theta}$$