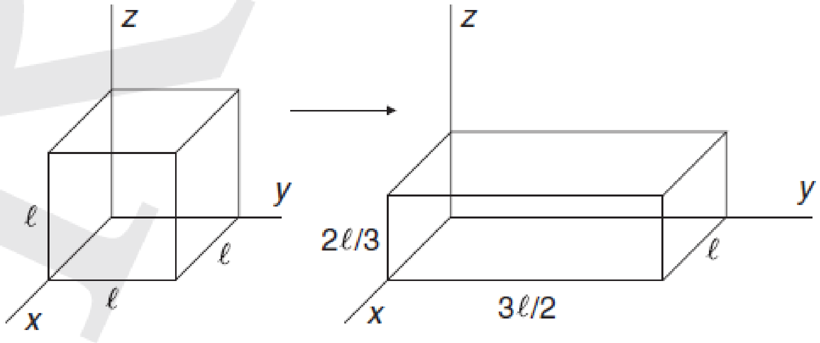
Ex 1 (similar to course 4 but we used B instead of Bd – we redo it here with Bd to illustrate compressible hyperelasticity)

An element of an incompressible material (in the reference state, a cube $\ell \times \ell \times \ell$) is placed in a Cartesian xyz-coordinate system as given in the figure below. Because of a load in the z-direction, the height of the element is reduced to $2\ell/3$. In the x-direction, the displacement is suppressed. The element can expand freely in the y-direction. The deformation is assumed to be homogeneous.

The material behaviour is described by a neo-Hookean relation, according to:

$$\underline{\sigma} = -p\underline{I} + G\underline{B}^{\mathrm{d}},$$

with $\underline{\sigma}$ the stress matrix, p the hydrostatic pressure (to be determined), $\underline{I}$ the unit matrix, G the shear modulus and $\underline{B}$ the left Cauchy–Green deformation matrix.



Determine the compressive force F_v in the z-direction that is necessary to realize this deformation.

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

$$B = FF^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/4 & 0 \\ 0 & 0 & 4/9 \end{bmatrix}$$

$$Tr(B) = 1 + \frac{9}{4} + \frac{4}{9} = \frac{36}{36} + \frac{81}{36} + \frac{16}{36} = 133/36$$

$$\frac{Tr(B)}{3} = \frac{133}{108} = 1.23$$

$$B^d = B - \frac{Tr(B)}{3} I = \begin{bmatrix} 1 - 1.23 & 0 & 0 \\ 0 & \frac{9}{4} - 1.23 & 0 \\ 0 & 0 & \frac{4}{9} - 1.23 \end{bmatrix} = \begin{bmatrix} -0.23 & 0 & 0 \\ 0 & 1.02 & 0 \\ 0 & 0 & -0.79 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} -p - 0.23G & 0 & 0 \\ 0 & -p + 1.02G & 0 \\ 0 & 0 & -p - 0.79G \end{bmatrix}$$

$$\sigma = \begin{bmatrix} ? & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F/S \end{bmatrix} = \begin{bmatrix} ? & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2F/3l^2 \end{bmatrix}$$

$$\sigma_{yy} = 0 \Rightarrow -p + 1.02G = 0 \Rightarrow p = 1.02G$$

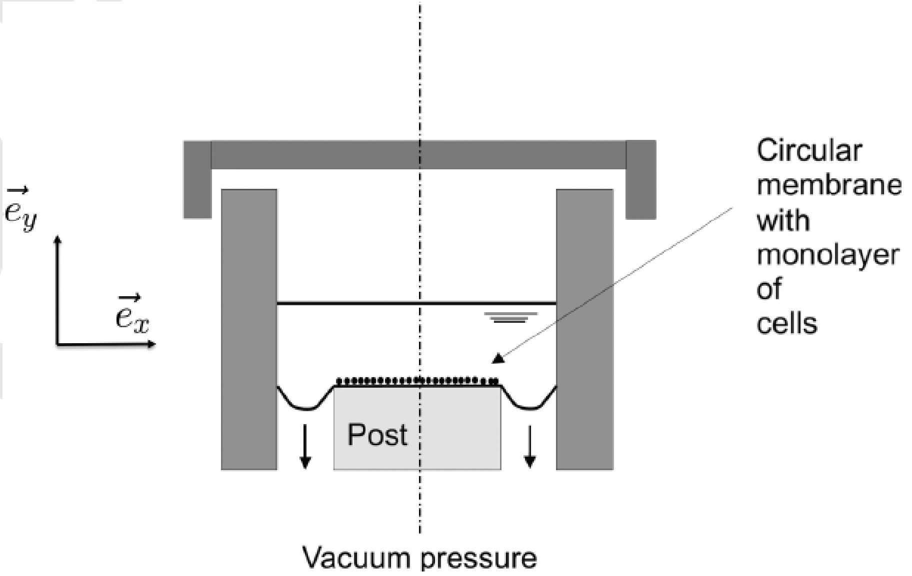
$$\sigma = \begin{bmatrix} -1.25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1.81 \end{bmatrix} G$$

$$\frac{2F}{3l^2} = -1.81G$$

$$F = -2.72Gl^2$$

Ex 2.

A popular device in tissue engineering is a cell stretching device, using membranes (see the figure). Cells are seeded on a circular membrane that rests on a cylindrically shaped post. On applying a vacuum pressure, the membrane is sucked into the gap between the post and the outer cylinder, and in this way the area with the monolayer of cells is equally stretched in the plane of the membrane (in this case the xy-plane). The layer can be considered to be in a state of **plane stress** ($\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$).



The deformation tensor for this state of strain can be given as:

$$\boldsymbol{F} = \lambda(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y) + \mu\vec{e}_z\vec{e}_z$$

with λ the stretch ratio in the x- and y-directions and μ the stretch ratio in the z-direction. The material behaviour of the membrane and the monolayer is modelled by:

$$\boldsymbol{\sigma} = \kappa(J-1) + G\boldsymbol{B}^{\mathrm{d}}$$

with κ the bulk modulus, G the shear modulus and $J = \det(\boldsymbol{F})$ and $\boldsymbol{B}^{\mathrm{d}} = \boldsymbol{B} - \frac{1}{3}\mathrm{tr}(\boldsymbol{B})\boldsymbol{I}$.

Derive an expression for μ as a function of λ, κ and μ.

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$$

$$B = FF^T = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \mu^2 \end{bmatrix}$$

$$Tr(B) = 2\lambda^2 + \mu^2$$

$$\frac{Tr(B)}{3} = \frac{2\lambda^2 + \mu^2}{3}$$

$$B^d = B - \frac{Tr(B)}{3} I = \begin{bmatrix} \lambda^2 - \frac{2\lambda^2 + \mu^2}{3} & 0 & 0 \\ 0 & \lambda^2 - \frac{2\lambda^2 + \mu^2}{3} & 0 \\ 0 & 0 & \mu^2 - \frac{2\lambda^2 + \mu^2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\lambda^2 + \mu^2}{3} & 0 & 0 \\ 0 & \frac{\lambda^2 + \mu^2}{3} & 0 \\ 0 & 0 & -\frac{2\lambda^2 - 2\mu^2}{3} \end{bmatrix}$$

$$B^d = \frac{1}{3} \begin{bmatrix} \lambda^2 - \mu^2 & 0 & 0 \\ 0 & \lambda^2 - \mu^2 & 0 \\ 0 & 0 & -2\lambda^2 + 2\mu^2 \end{bmatrix}$$

$$J = \mu\lambda^2$$

$$\sigma = \kappa \begin{bmatrix} J-1 & 0 & 0 \\ 0 & J-1 & 0 \\ 0 & 0 & J-1 \end{bmatrix} + G/3 \begin{bmatrix} \lambda^2 - \mu^2 & 0 & 0 \\ 0 & \lambda^2 - \mu^2 & 0 \\ 0 & 0 & -2\lambda^2 + 2\mu^2 \end{bmatrix}$$

$$\sigma_{zz} = 0 \Rightarrow \kappa(\mu\lambda^2 - 1) + G(-2\lambda^2 + 2\mu^2)/3 = 0$$

$$\frac{2G}{3}\mu^2 + \kappa\lambda^2\mu - \left(\frac{2G}{3}\lambda^2 + \kappa\right) = 0$$

$$\mu^2 + \frac{3\kappa}{2G}\lambda^2\mu - \left(\lambda^2 + \frac{3\kappa}{2G}\right) = 0$$

$$\gamma = \frac{3\kappa}{2G}$$

$$\mu^2 + \gamma\lambda^2\mu - (\lambda^2 + \gamma) = 0$$

$$\Delta = \gamma^2\lambda^4 + 4(\lambda^2 + \gamma)$$

$$\mu = \frac{-\gamma\lambda^2 + \sqrt{\Delta}}{2} = \frac{-\gamma\lambda^2 + \sqrt{\gamma^2\lambda^4 + 4(\lambda^2 + \gamma)}}{2}$$

$$\mu = \frac{3\kappa}{2G}\lambda^2 \frac{-1 + \sqrt{1 + 4(1/\gamma^2\lambda^2 + 1/\gamma\lambda^4)}}{2}$$

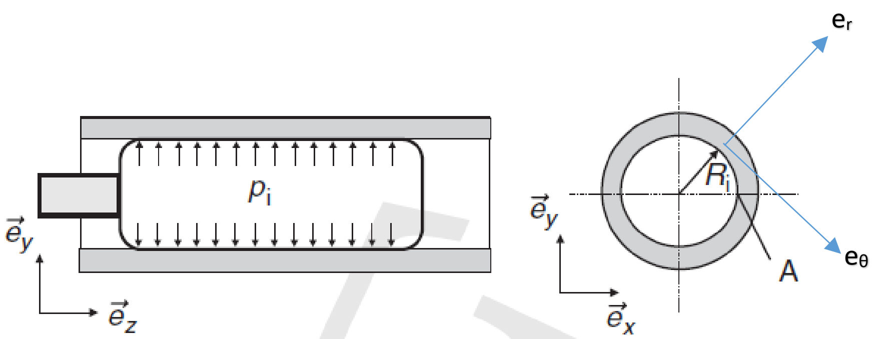
Poisson effect in small strain

$\lambda = 1 + \varepsilon$ with $\varepsilon \ll 1$

https://en.wikipedia.org/wiki/Young%27s_modulus

Ex 3.

During a percutaneous angioplasty, the vessel wall is expanded by inflating a balloon. At a certain moment during inflation, the internal pressure $p_i = 0.2$ [MPa], and the internal radius R_i of the vessel is increased by 10%. Assume that the length of the balloon and the length of the vessel wall in contact with the balloon do not change. Further, it is known that the wall behaviour can be modelled according to Hooke's law, with Young's modulus of the wall $E = 8$ [MPa] and the Poisson's ratio of the wall $\nu = 1/3$.



(a) Calculate the strain in the circumferential direction at the inner side of the wall.

(b) Calculate the strain in the $\vec{e}_x$-direction at point A at the inner side of the vessel wall.

(c) Calculate the stress in circumferential direction at point A.

(d) for course 6 -> Repeat the same exercise with a non compressible NeoHookean model (parameter G) and a compressible NeoHookan model (parameters G and K).

a)

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}_{xyz} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix}_{r\theta z}$$

b)

$$\varepsilon_{\theta\theta} = \frac{2\pi(r_i - R_i)}{2\pi R_i} = \frac{(r_i - R_i)}{R_i} = 10\%$$

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \nu(\varepsilon_{\theta\theta} + \varepsilon_{zz})$$

$$\varepsilon_{rr} = -\frac{p}{E} - \nu\varepsilon_{\theta\theta}$$

$$\varepsilon_{rr} = -\frac{0.1}{4} - \frac{0.1}{3} = -\frac{0.7}{12} = 0.06$$

c)

$$\sigma = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$$

$$\sigma_{\theta\theta} = E\varepsilon_{\theta\theta}$$