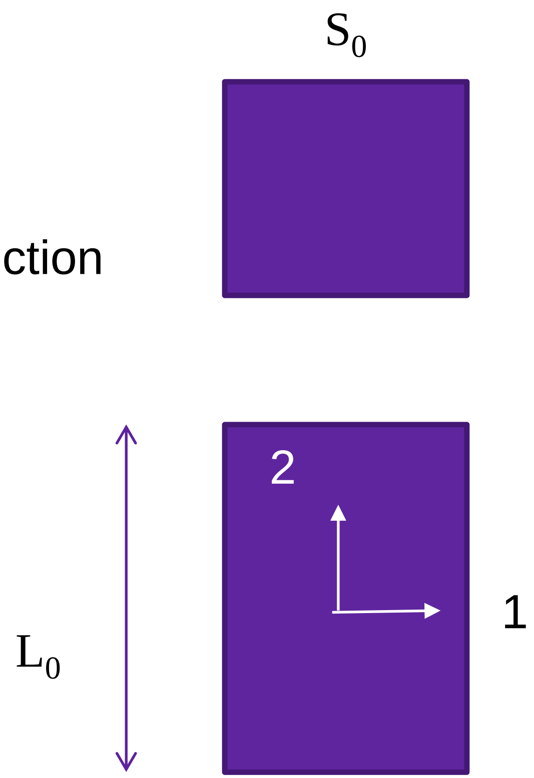
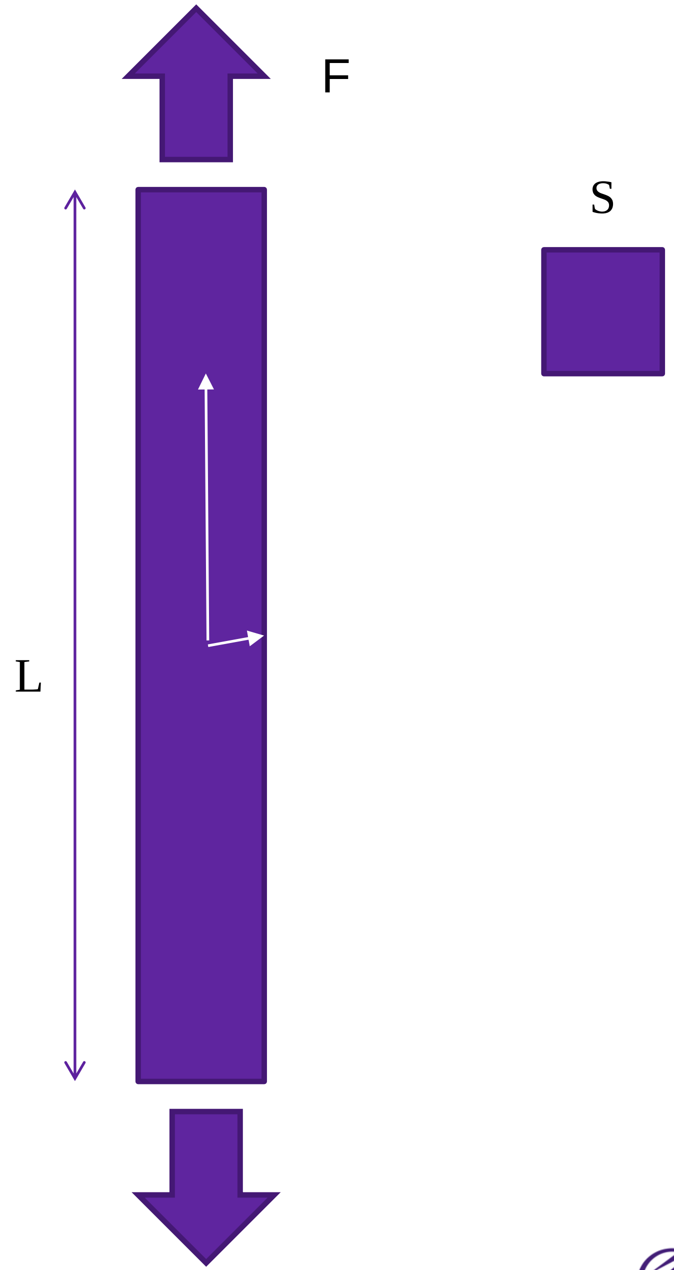
UNIAXIAL TENSION

Cross section

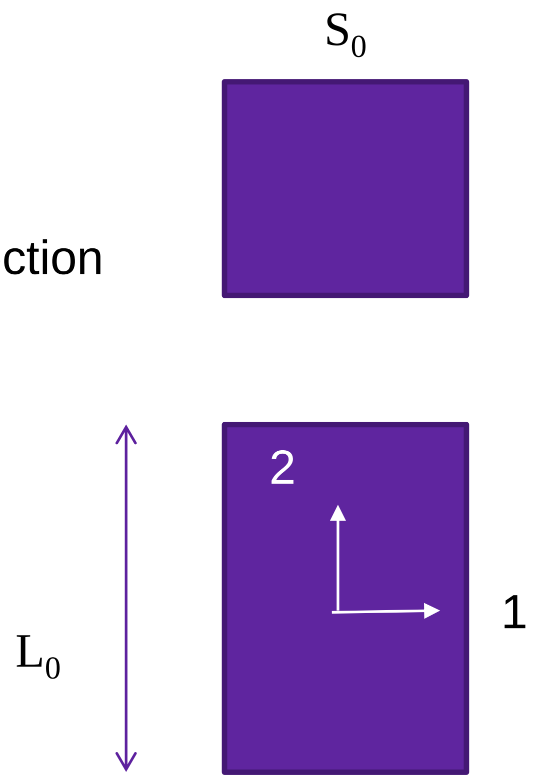


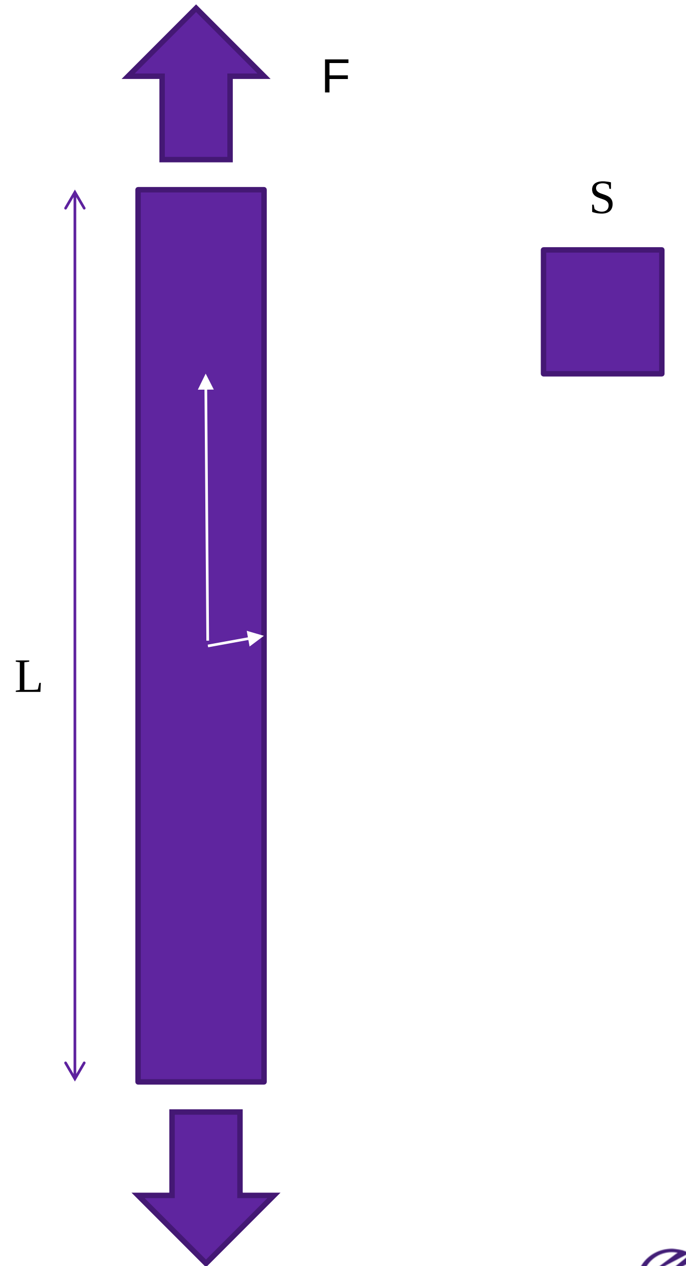


$$F = \begin{bmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}$$

UNIAXIAL TENSION

Cross section

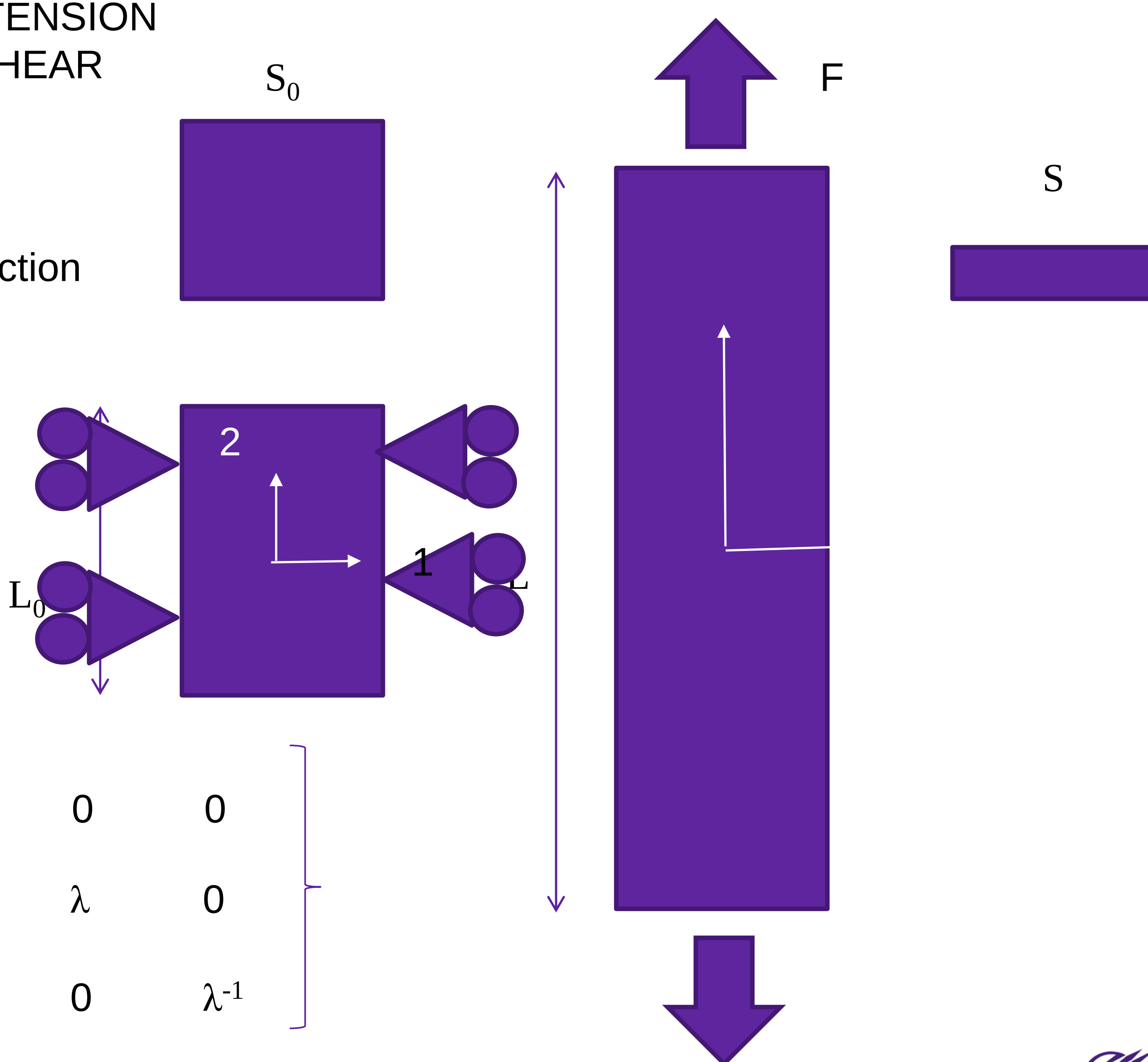




$$C = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{2} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

PLANAR TENSION = PURE SHEAR

Cross section

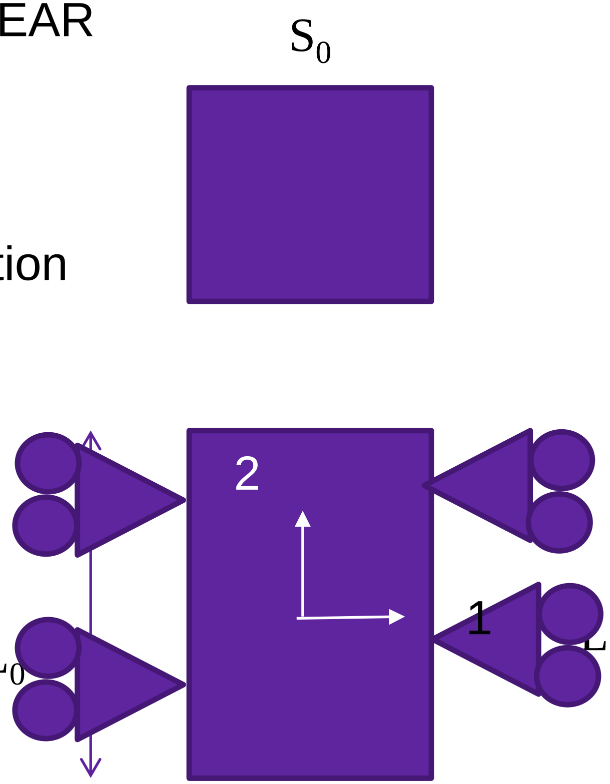


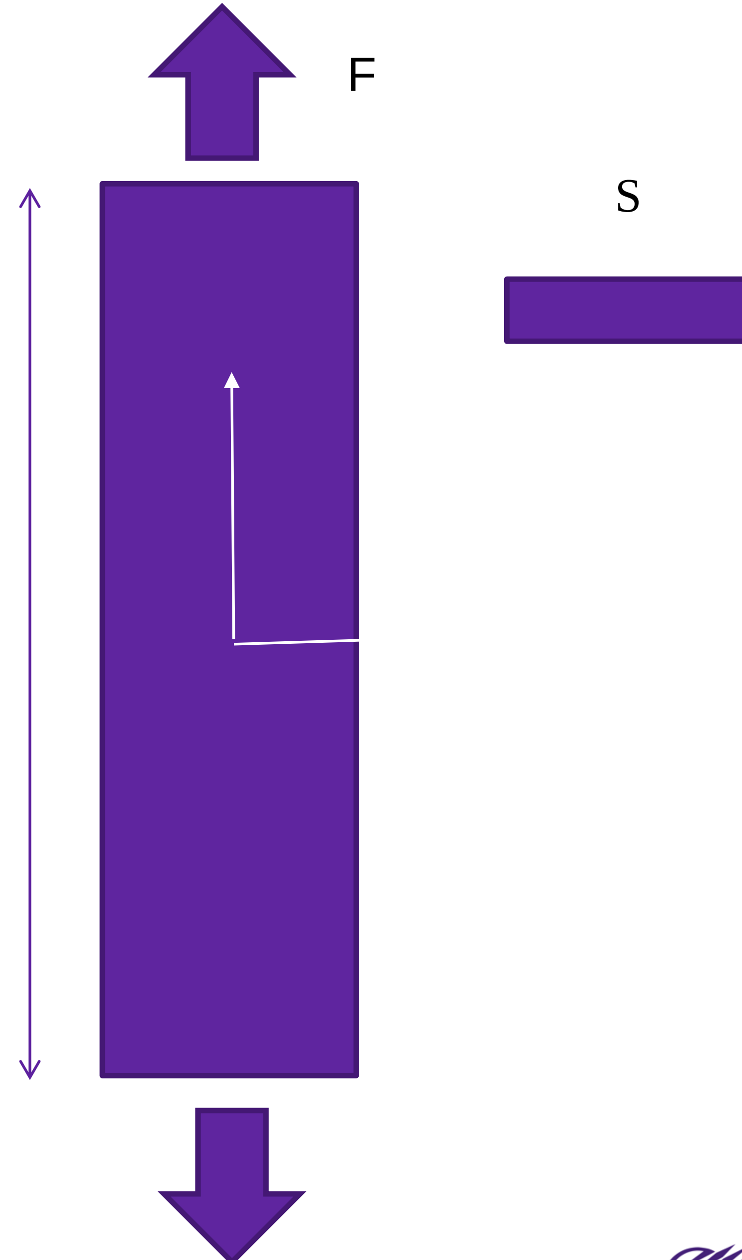
$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

avril@emse.fr

PLANAR TENSION
= PURE SHEAR

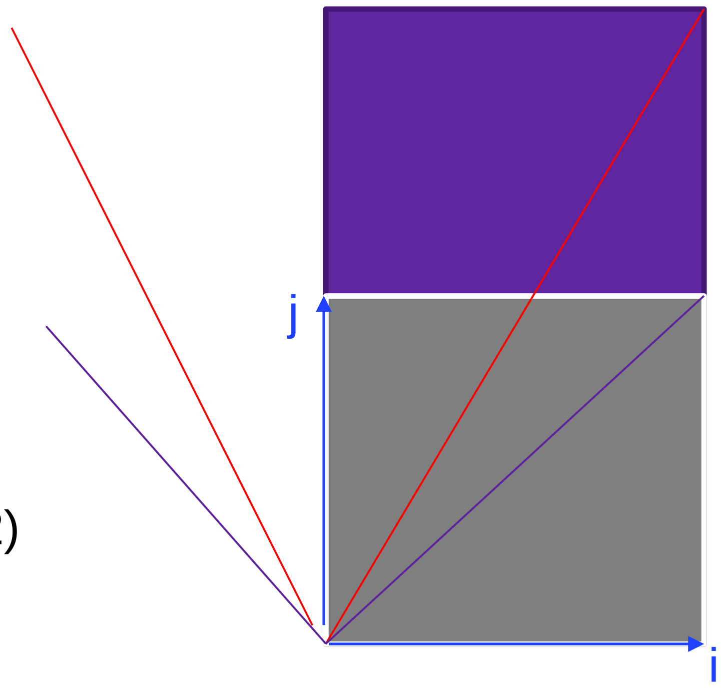
Cross section





$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

PLANAR TENSION = PURE SHEAR



Cross section

||E1|| = Lo sqrt(2)
||e1|| = Lo sqrt(1+lambda)

||e1||/||E1|| = sqrt([1+lambda]/2)
pythagorus

e1 = a E1 + b E2|

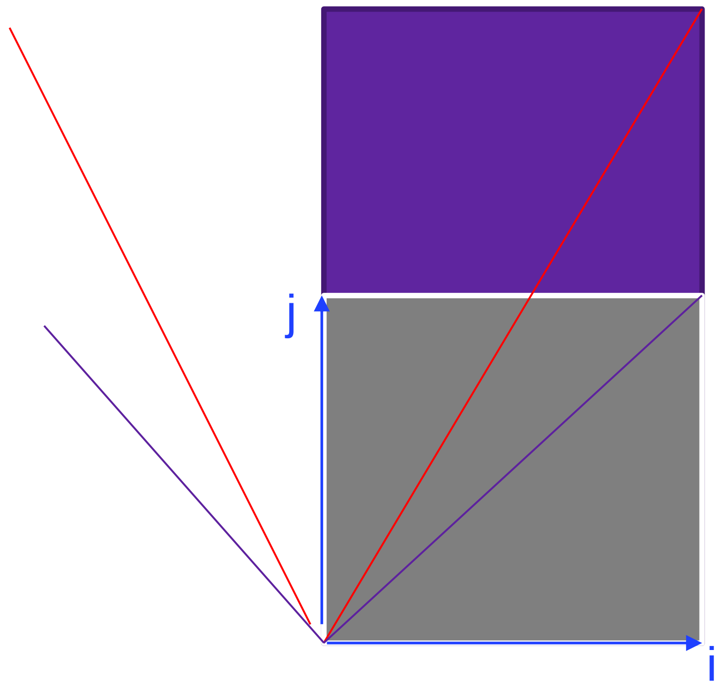
e2 = c E1 + d E2

J =[(1+λ)^2-(-1+λ)^2]μ=λμ

$$F = \begin{bmatrix} (1+\lambda)/2 & (-1+\lambda)/2 & 0 \\ (-1+\lambda)/2 & (1+\lambda)/2 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

a= e1.E1=(i+λj).(i+j)=1+λ
b= e1.E2=(i+λj).(-i+j)=-1+λ
c= e2.E1=(-i+λj).(i+j)=-1+λ
d= e2.E2=(-i+λj).(-i+j)=-1+λ

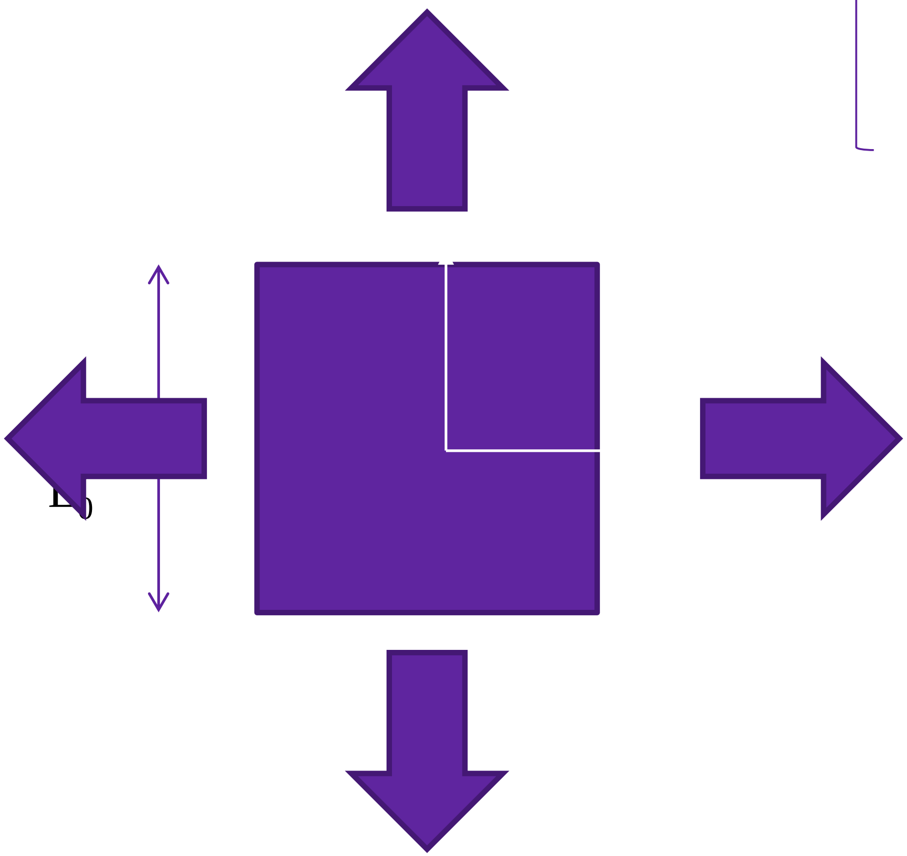
PLANAR TENSION
= PURE SHEAR



$$C = \begin{bmatrix} (1+\lambda^2)/2 & (-1+\lambda^2)/2 & 0 \\ (-1+\lambda^2)/2 & (1+\lambda^2)/2 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

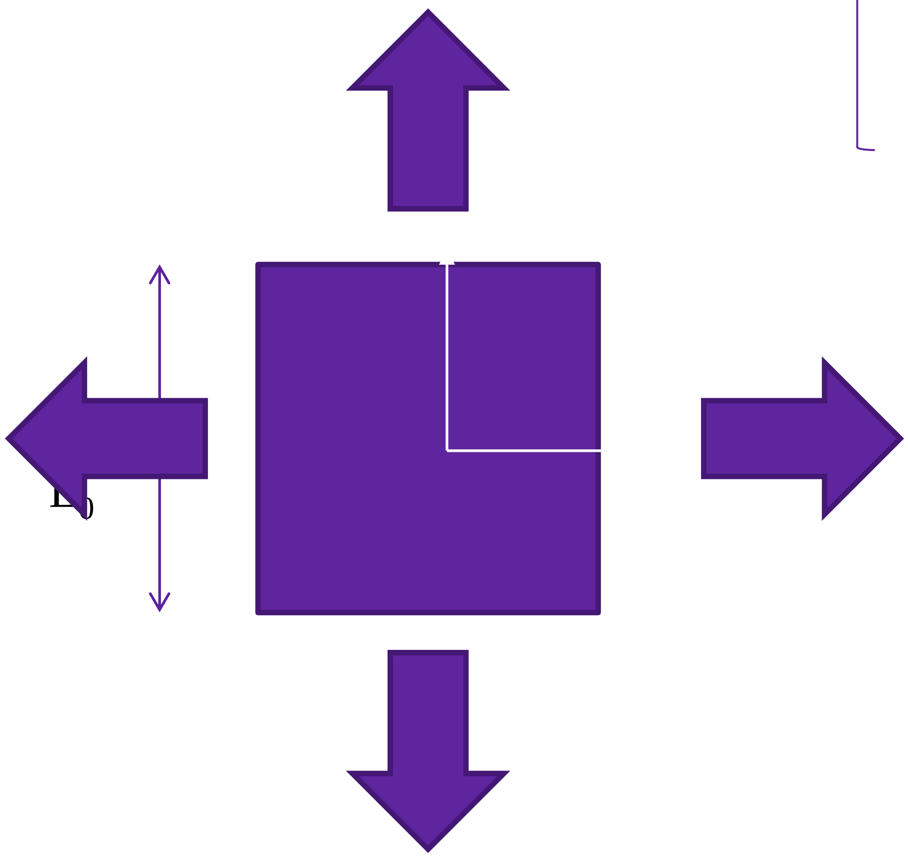
$$E = \begin{bmatrix} (-1+\lambda^2)/4 & (-1+\lambda^2)/4 & 0 \\ (-1+\lambda^2)/4 & (-1+\lambda^2)/4 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$

$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$$



EQUIBIAXIAL TENSION

$$C = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-4} \end{bmatrix}$$

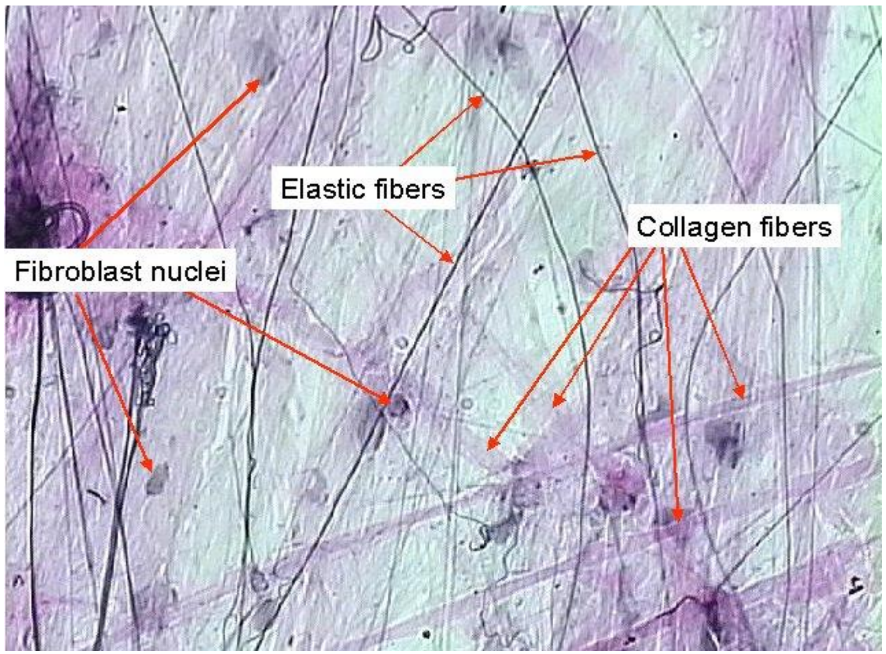


EQUIBIAXIAL TENSION

avril@emse.fr

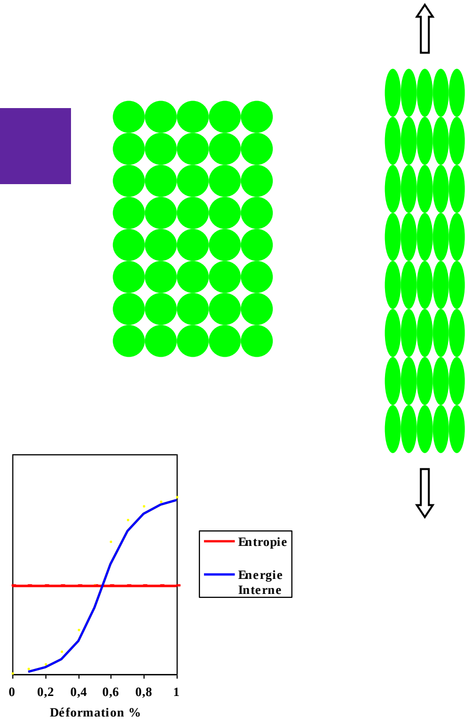
$$(Fa).(b) = (a).(F^T b)$$

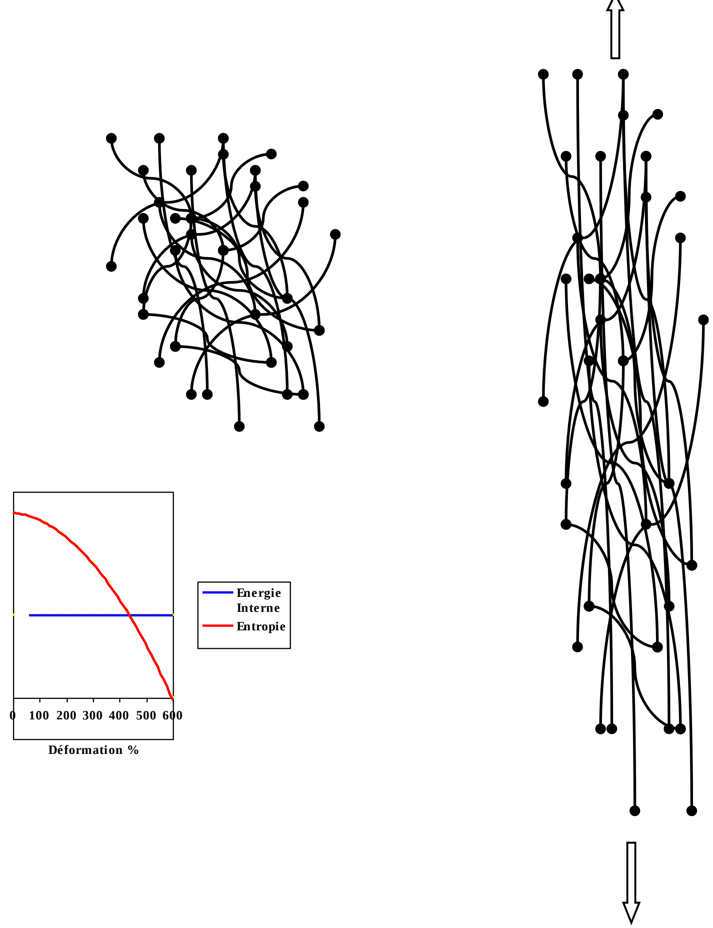
Soft biological tissues: many challenges for continuum mechanics



avril@emse.fr

MINES Saint-Étienne





enthalpic

elasticity

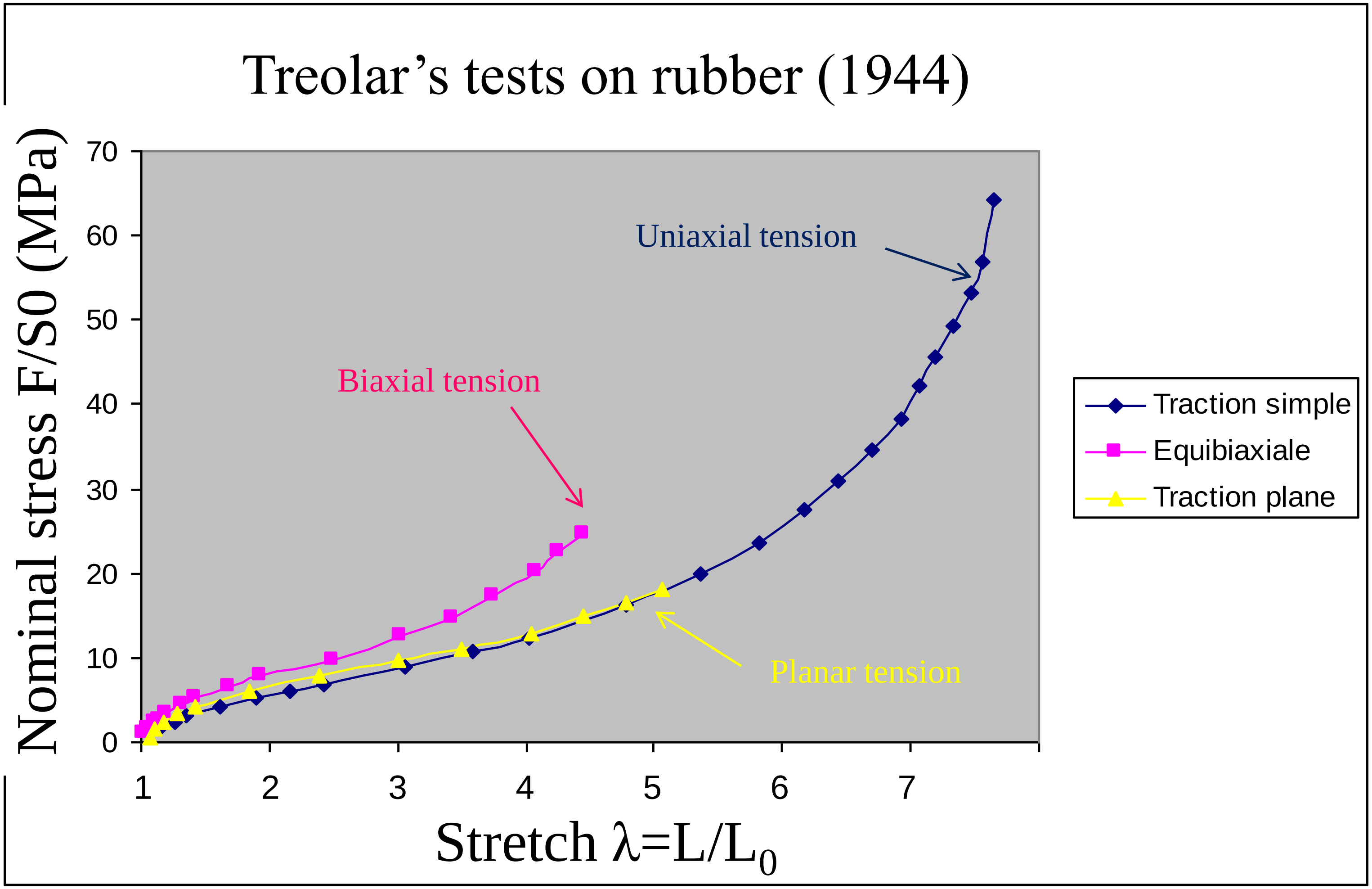
(cristal)

entropic

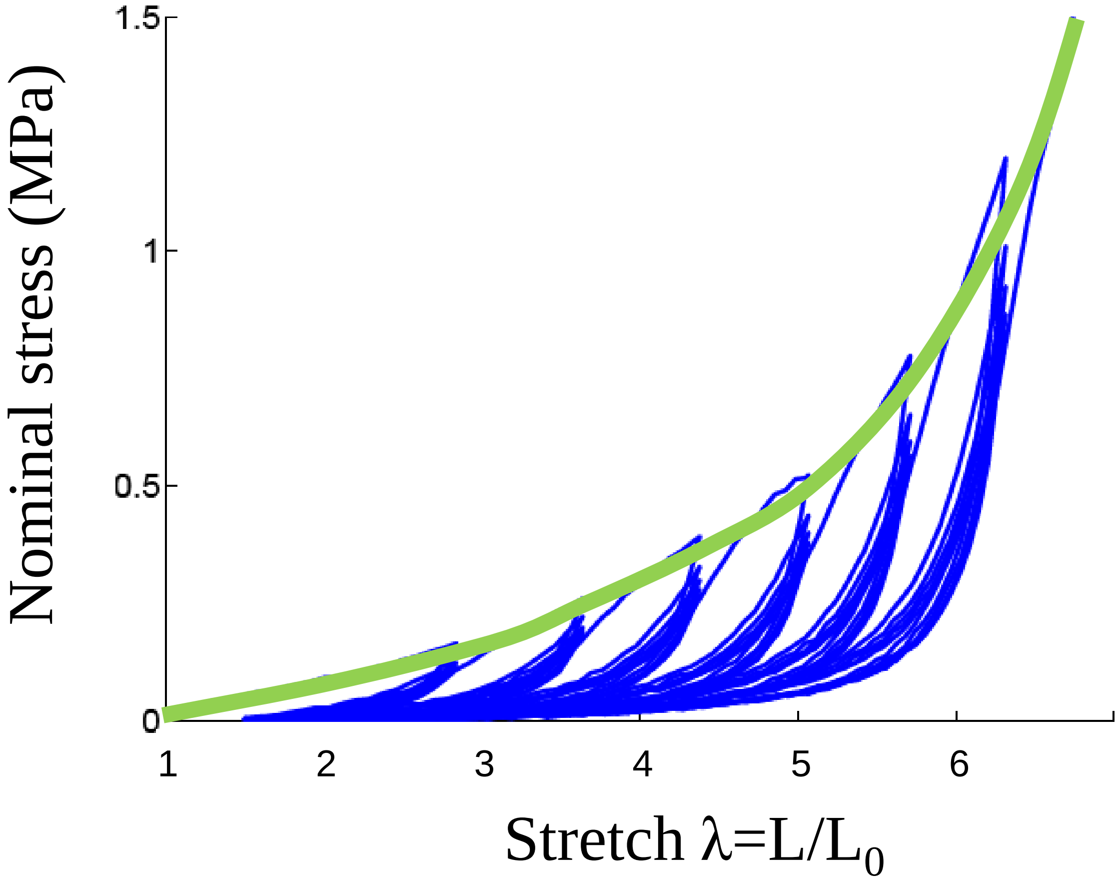
elasticity

(biological tissue)

Hyperelasticity



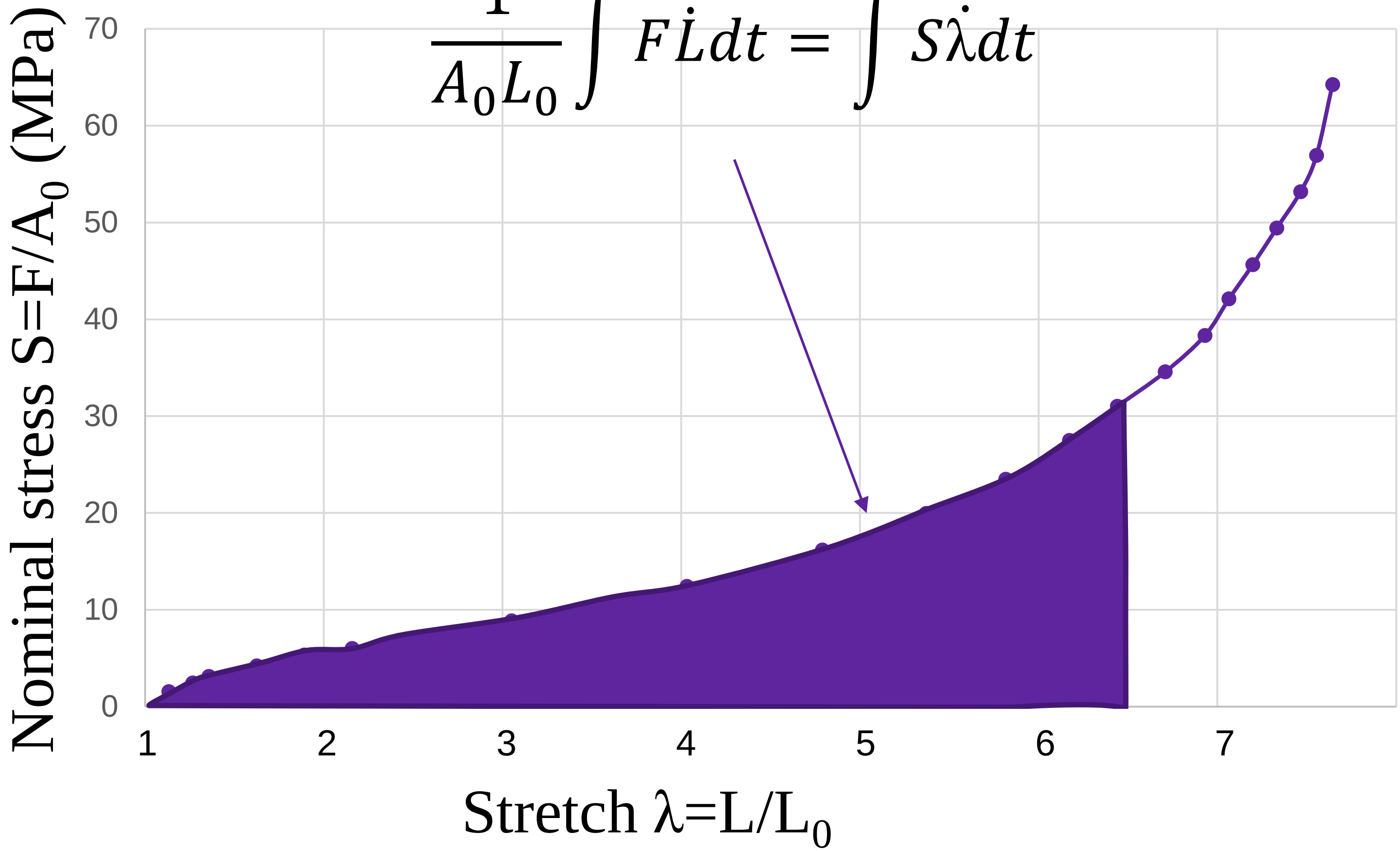
Pseudo-hyperelasticity: Irreversible effects are neglected…



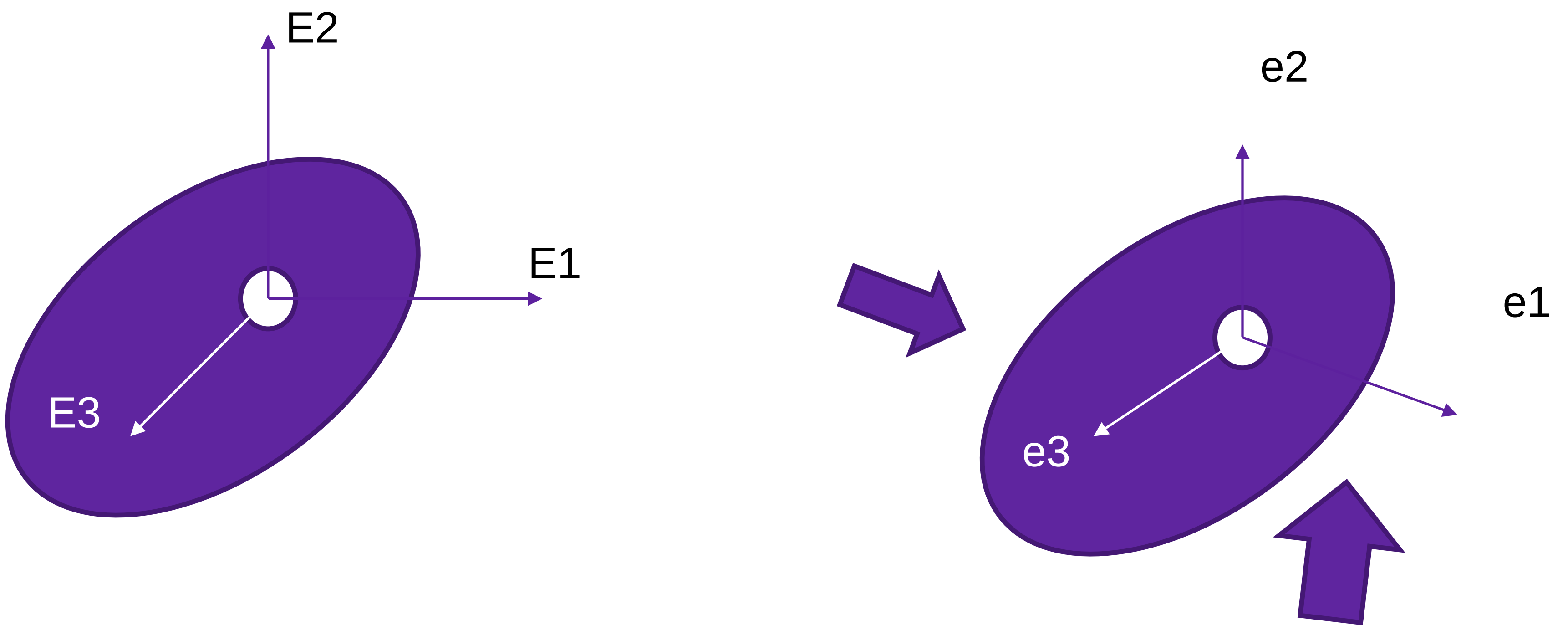
Strain energy density

Stored energy per unit volume

$$\frac{1}{A_0 L_0}\int F\dot{L}dt = \int S\dot{\lambda}dt$$



KINEMATICS



Reference (initial)

Deformed

Deformation gradient F

F E1 = e1
F E2 = e2
F E3 = e3
F Ei = ei

E1 = (1,0,0)
E2 = (0,1,0)
E3 = (0,0,1)

$$\begin{Bmatrix} E11 \\ E12 \\ E13 \end{Bmatrix}$$

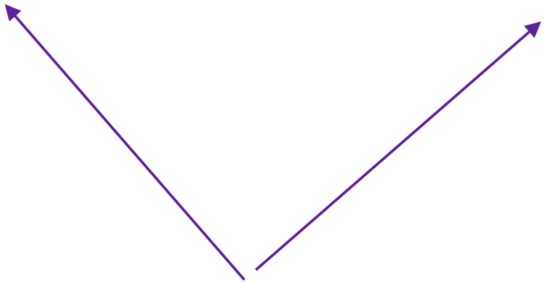
J=det(F) => change of volume

$$F = \begin{Bmatrix} e11 & e21 & e31 \\ e12 & e22 & e32 \\ e13 & e23 & e33 \end{Bmatrix} = \begin{Bmatrix} e11 \\ e12 \\ e13 \end{Bmatrix}$$

$$F = \begin{bmatrix} F11 & F12 & F13 \\ F21 & F22 & F23 \\ F31 & F32 & F33 \end{bmatrix}$$

$$F^T = \begin{bmatrix} F11 & F21 & F31 \\ F12 & F22 & F32 \\ F13 & F23 & F33 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$F^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = 0$$

avril@emse.fr

C = cauchy green stretch tensor

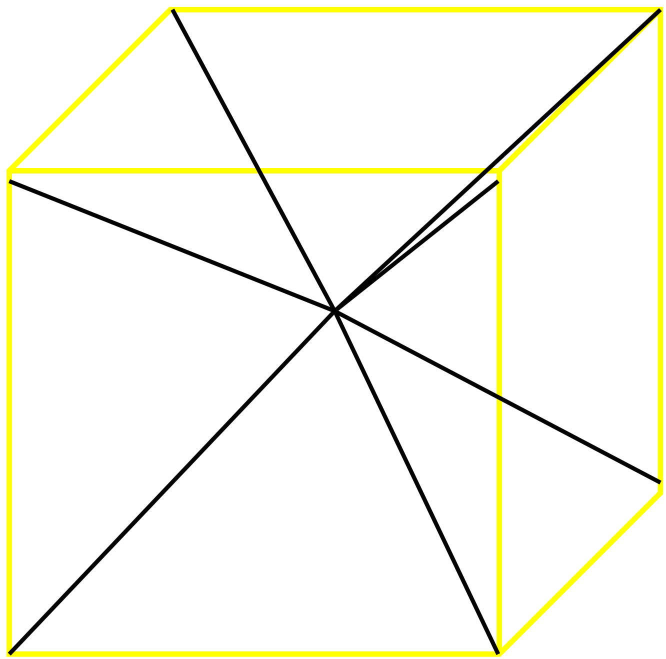
first invariant of $C=F^TF$

Pythogorus

Change of length^2 of diagonals

= Tr(C)

= C11 + C22 + C33



$L_0^2 = 3a^2$

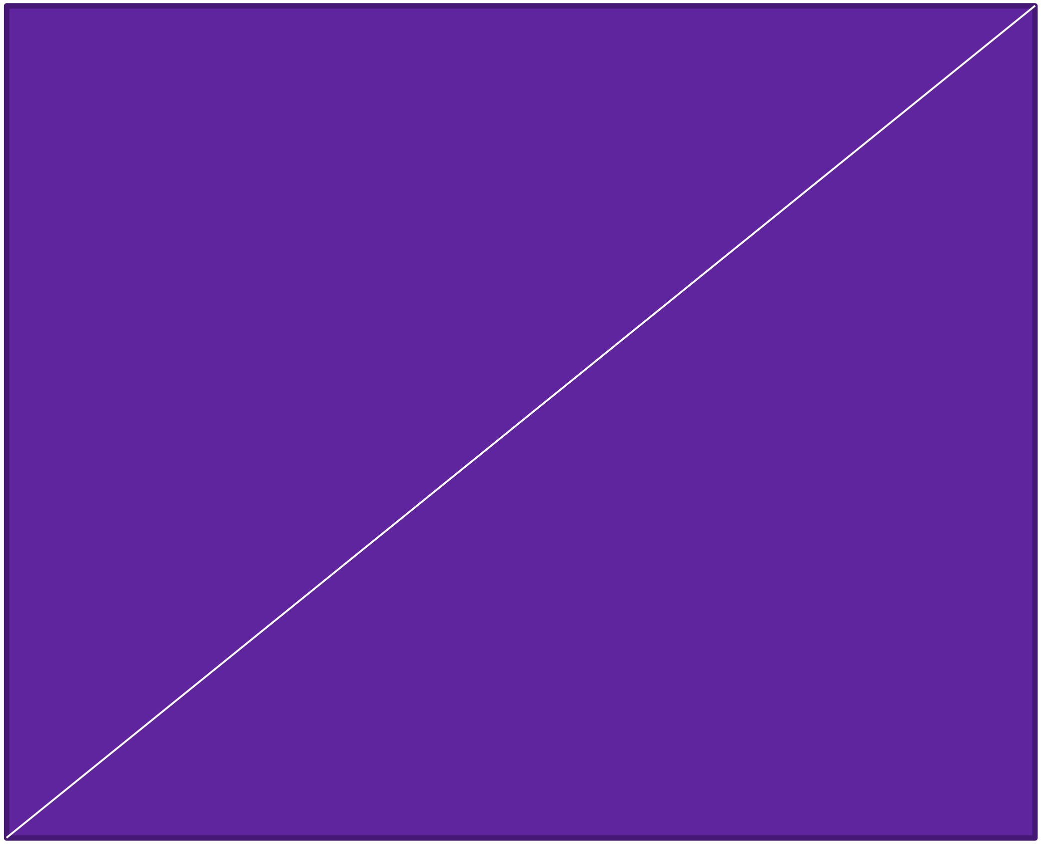
$L^2 = (\lambda_1 a)^2 + (\lambda_2 a)^2 + (\lambda_3 a)^2$

L^2 / L_0^2

$= [(\lambda_1)^2 + (\lambda_2)^2 + (\lambda_3)^2] / 3$

$= tr(C) / 3$

$I_1 = tr(C)$



Neo Hooke behaviour

Energy = G/2 (I1 – 3)

Energy of a spring = ½ k x^2

avril@emse.fr

Deformation mapping in 3D

$$y_i = x_i + u_i(x_1, x_2, x_3, t)$$

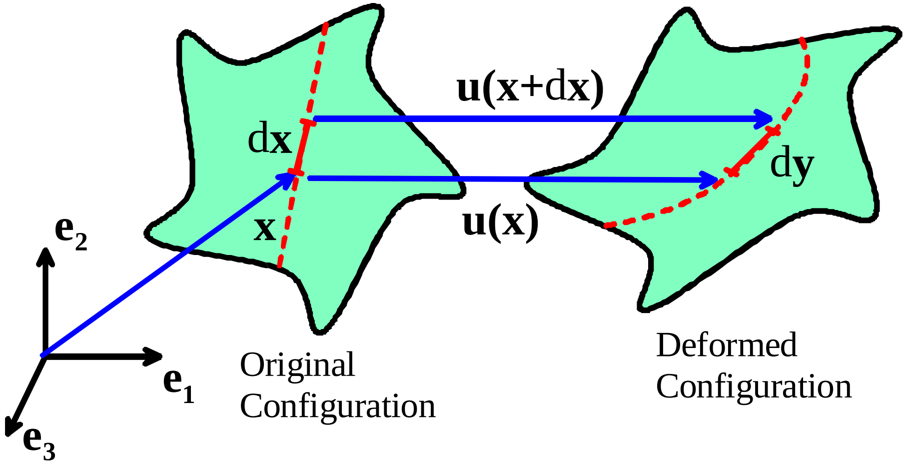
Deformation Gradient

$$\nabla \mathbf{y} = \nabla\left(\mathbf{x} + \mathbf{u}(\mathbf{x})\right) = \mathbf{F}$$

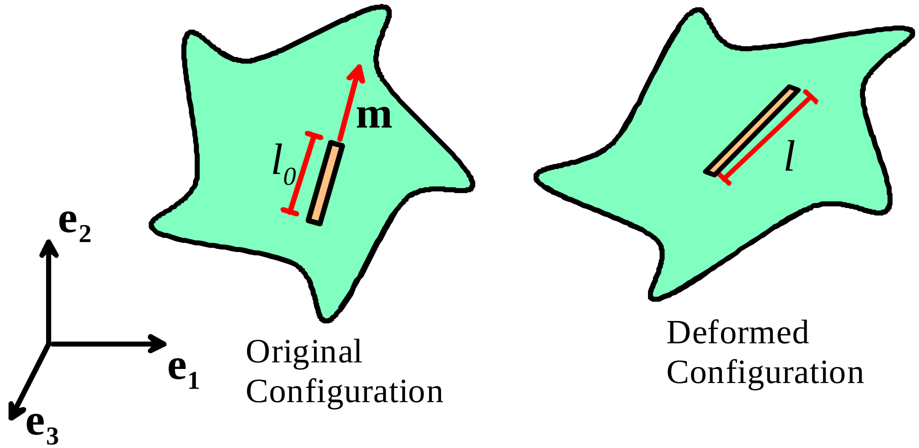
or $$\frac{\partial y_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(x_i + u_i\right) = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}$$

$$d\mathbf{y} = \mathbf{F} \cdot d\mathbf{x}$$

$$dy_i = F_{ik} dx_k$$



Green Lagrange strain



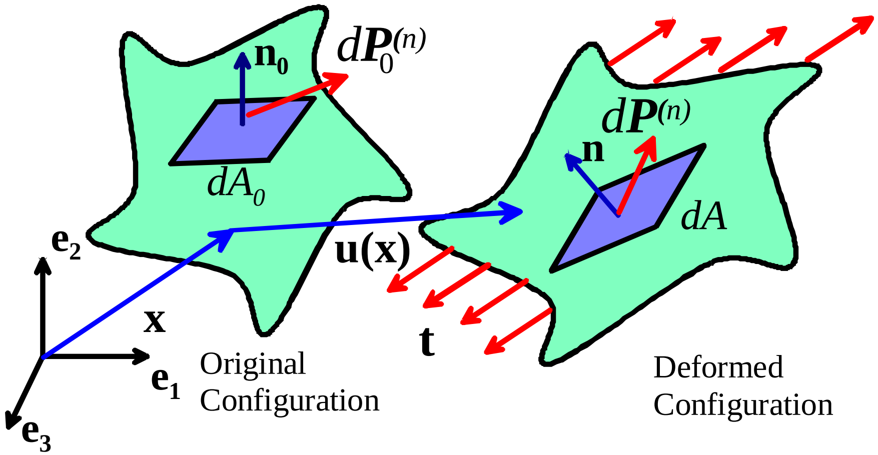
$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \quad \text{or} \quad E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$$

$$\mathbf{m} \cdot \mathbf{E} \cdot \mathbf{m} = E_{ij} m_i m_j = \frac{l^2 - {l_0}^2}{2{l_0}^2} = \frac{\delta l}{l_0} + \frac{(\delta l)^2}{2l_0^2}$$

Stress measures

True / Cauchy

$$\boldsymbol{\sigma}$$



Nominal/ 1st Piola-Kirchhoff

$$\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \qquad S_{ij} = JF_{ik}^{-1}\sigma_{kj}$$

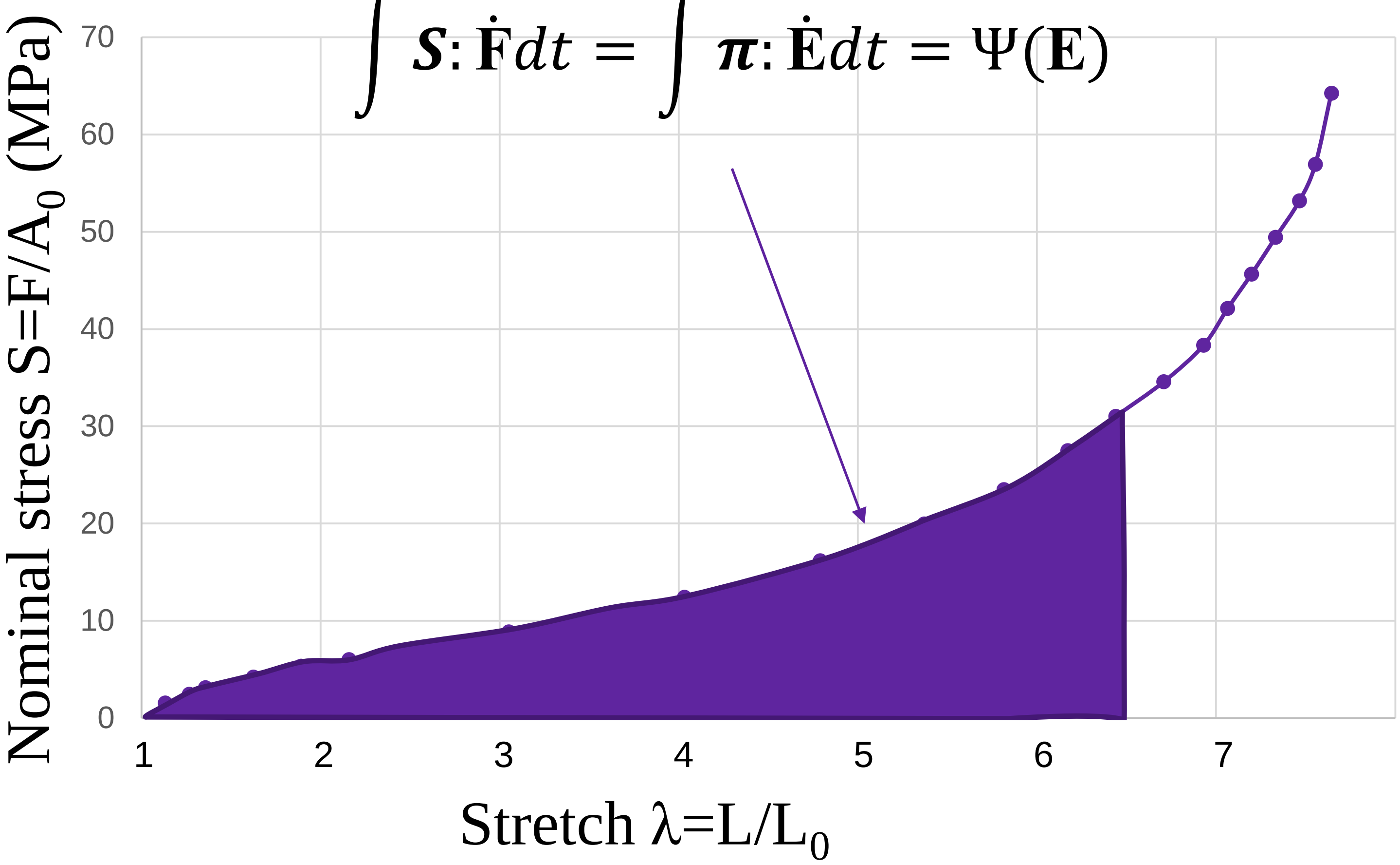
Material/2nd Piola-Kirchhoff

$$\boldsymbol{\pi} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \quad \pi_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$$

Hyperelasticity

Stored energy per unit volume

$$\int \boldsymbol{S}:\dot{\mathbf{F}}dt = \int \boldsymbol{\pi}:\dot{\mathbf{E}}dt = \Psi(\mathbf{E})$$



avril@emse.fr

compressible hyperelastic behaviour

$$\boldsymbol{\sigma} = J\mathbf{F}.\frac{\partial \Psi}{\partial \mathbf{E}}.\mathbf{F}^T$$

incompressible hyperelastic behaviour (J=1)

$$\boldsymbol{\sigma} = \mathbf{F}.\frac{\partial \Psi}{\partial \mathbf{E}}.\mathbf{F}^T + c\mathbf{I}$$

Strain energy density:

$$\Psi = ?$$