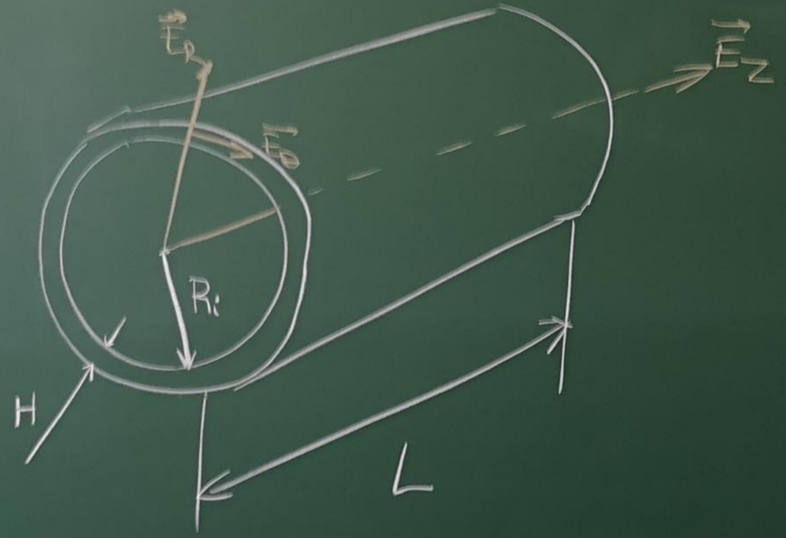
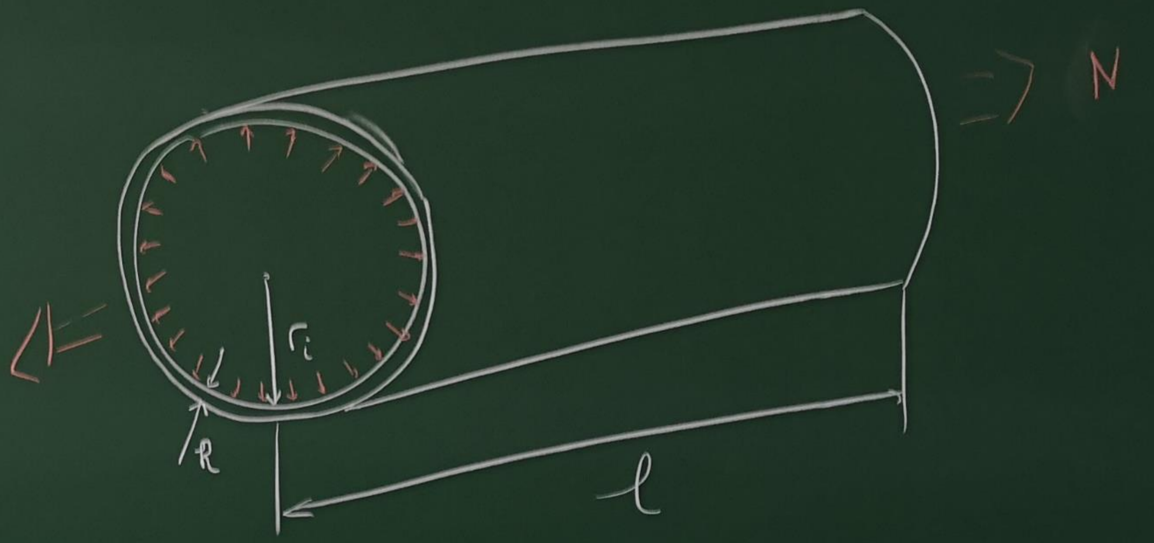
Cylinder modeling an artery



REFERENCE CONFIG. = NO LOAD



DEFORMED CONFIG.

Thin wall. $\frac{H}{R_i} \ll 1$ $\frac{R_i}{L} \ll 1$ (large)

↳ We neglect variations across the thickness

everything uniform.

1) Write the deformation gradient F as a function of $\lambda = \frac{r_i}{R_i}$ and $\lambda_z = \frac{\ell}{L}$ assuming incompressibility

2) Write the Cauchy stress using equilibrium (as a function of p and N)

3) Assume the material is Neo Hookean. $\Psi = \frac{G}{2}(I_1 - 3)$

Write the Cauchy stress as a function of G, λ, λ_z

4) Write p and N as functions of G, λ, λ_z

1)

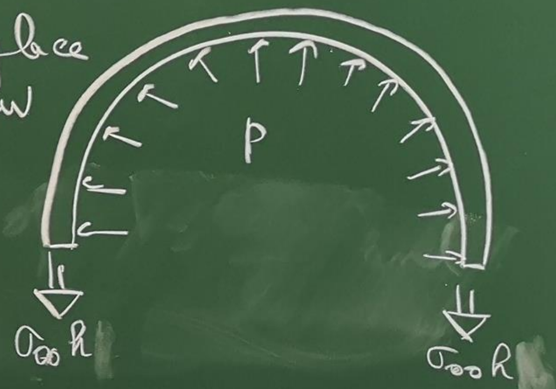
$$F = \begin{bmatrix} \frac{1}{\lambda\lambda_z} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_z \end{bmatrix} \qquad J = \det(F) = 1$$

2) approx. for thin wall

$$(\vec{E}_R, \vec{E}_\theta, \vec{E}_z)$$

Laplace law

$$\sigma = \begin{bmatrix} \sim 0 & 0 & 0 \\ 0 & p\frac{r_i}{h} & 0 \\ 0 & 0 & \frac{N}{2\pi r_i h} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \boxed{\lambda_z\lambda^2 p \frac{R_i}{H}} & 0 \\ 0 & 0 & \boxed{\lambda_z \frac{N}{2\pi R_i H}} \end{bmatrix}$$



3)

$$c = -\frac{G}{\lambda^2\lambda_z^2}$$

$$\boxed{\sigma = G F^{T} F + c\,I}$$

$$\sigma = \begin{bmatrix} \boxed{\frac{G}{\lambda^2\lambda_z^2} + c} & 0 & 0 \\ 0 & G\lambda^2 + c & 0 \\ 0 & 0 & G\lambda_z^2 + c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \boxed{G\left(\lambda^2 - \frac{1}{\lambda^2\lambda_z^2}\right)} & 0 \\ 0 & 0 & \boxed{G\left(\lambda_z^2 - \frac{1}{\lambda^2\lambda_z^2}\right)} \end{bmatrix}$$

4)

$$p = \frac{GH}{R_i}\left[\frac{1}{\lambda_z} - \frac{1}{\lambda^4\lambda_z^3}\right]$$

$$N = 2\pi G R_i H\left[\lambda_z - \frac{1}{\lambda^2\lambda_z^3}\right]$$

$$\lambda_z = 1 \Rightarrow P = \frac{GH}{R_i}\left(1 - \frac{1}{\lambda^4}\right)$$

$$\lambda_z = 1.2 \Rightarrow P = \frac{GH}{R_i\, 1.2^3}\left(1.2^2 - \frac{1}{\lambda^4}\right)$$

$H = 2\,mm$

$R_i = 20\,mm \Rightarrow \frac{H}{R_i} = 0.1$

$G = 200\,kPa$ (Rubber)

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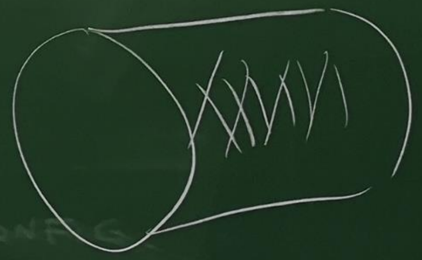
$$\lambda_z = 1.2 \Rightarrow p = \frac{GH}{R_i\, 1.2^3}\left(1.2^2 - \frac{1}{\lambda^4}\right)$$

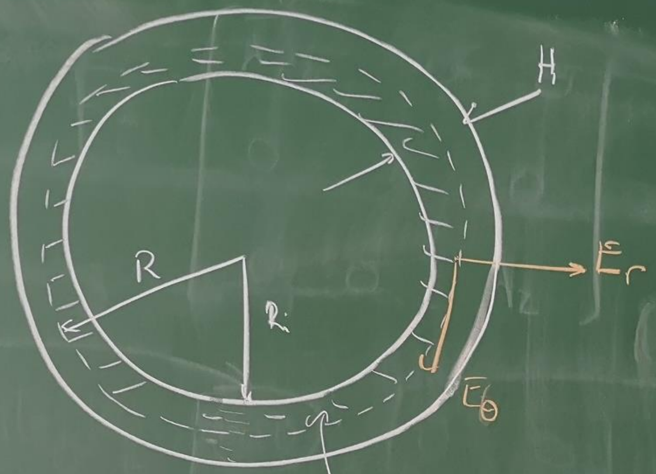
$H = 2\ \text{mm}$

$R_i = 20\ \text{mm} \Rightarrow \frac{H}{R_i} = 0.1$

$G = 200\ \text{kPa}$ (Rubber)

$100\ \text{mmHg} = 13\ \text{kPa}$





$$F(R) = \begin{bmatrix} \lambda_R(R) & 0 & 0 \\ 0 & \lambda(R) & 0 \\ 0 & 0 & \lambda_z \end{bmatrix}$$

$$\sigma(R) = \begin{bmatrix} \sigma_{rr}(r) & 0 & 0 \\ 0 & \sigma_{\theta\theta}(r) & 0 \\ 0 & 0 & \sigma_{zz}(r) \end{bmatrix}$$

$$\lambda_i = \frac{r_i}{R_i}$$

$$V = L\pi(R^2 - R_i^2)$$ reference conf.

$$v = \ell\pi(r^2 - r_i^2)$$ deformed conf.

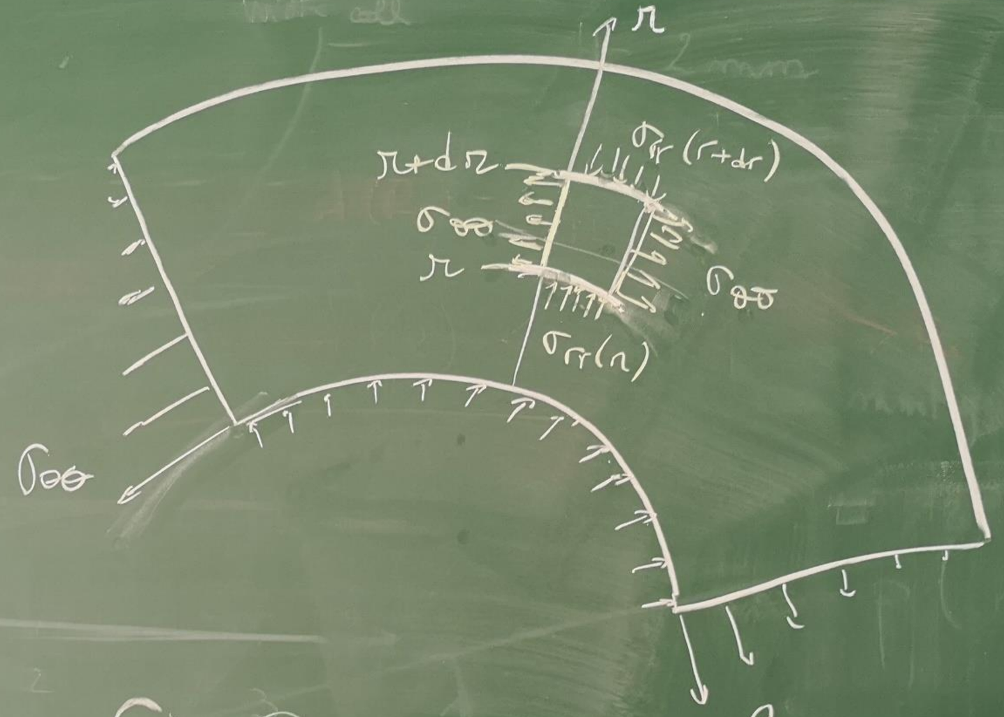
$$v = V \Rightarrow \lambda_z(r^2 - r_i^2) = R^2 - R_i^2$$

$$\Rightarrow \lambda^2 R^2 - \lambda_i^2 R_i^2 = (R^2 - R_i^2)/\lambda_z$$

$$\Rightarrow \lambda^2 = \frac{1}{\lambda_z} - \left(\frac{1}{\lambda_z} - \lambda_i^2\right)\frac{R_i^2}{R^2} \Rightarrow \lambda = \sqrt{\frac{1}{\lambda_z} - \left(\frac{1}{\lambda_z} - \lambda_i^2\right)\frac{R_i^2}{R^2}}$$

$$B(R) = \begin{bmatrix} \dfrac{1}{\lambda_2^2\left[\dfrac{1}{\lambda_2} - \left(\dfrac{1}{\lambda_2} - \lambda_1^2\right)\dfrac{R_i^2}{R^2}\right]} & 0 & 0 \\ 0 & \dfrac{1}{\lambda_2} - \left(\dfrac{1}{\lambda_2} - \lambda_1^2\right)\dfrac{R_i^2}{R^2} & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix}$$

$$B = F^T F$$



$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r} = 0$$

$$\int_{r_i}^{r_o} \frac{\partial \sigma_{rr}}{\partial r}\, dr = \sigma_{rr}(r_o) - \sigma_{rr}(r_i) = -p$$

extension of Laplace law for local eq.

$$p = \int_{r_i}^{r_o} \frac{\sigma_{\theta\theta} - \sigma_{rr}}{r}\, dr$$

$$\lambda = \sqrt{\frac{1}{\lambda z} - \left(\frac{1}{\lambda z} - \lambda_i^{2}\right)\frac{R_i^{2}}{R^{2}}}$$

$$\lambda' = \ldots$$

$$p = G \int_{r_i}^{r_o} \left\{ \frac{\left[\frac{1}{\lambda_z} - \left[\frac{1}{\lambda_z} - \lambda_i^2\right] \frac{R_i^2}{R^2}\right]}{r} - \frac{1}{r \lambda_z^2 \left(\frac{1}{\lambda_z} - \left[\frac{1}{\lambda_z} - \lambda_i^2\right] \frac{R_i^2}{R^2}\right)} \right\} dr$$

$$r = \lambda R$$

$$dr = \lambda \, dR + \frac{\partial \lambda}{\partial R} R \, dR$$

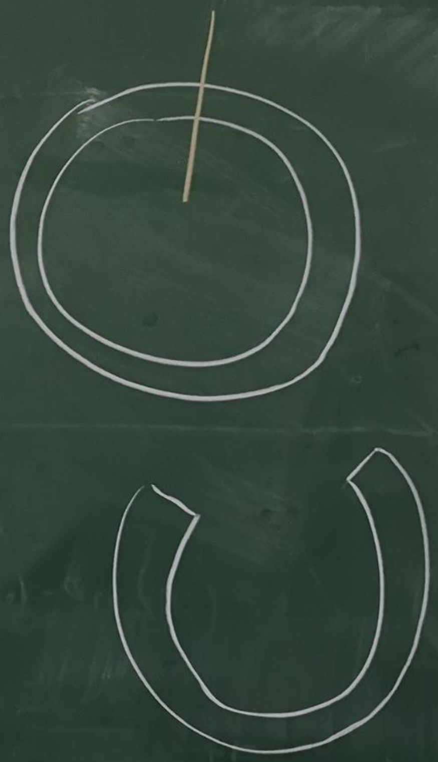
$$p = G \int_{R_i}^{R_o} \left[\frac{\lambda}{\lambda R} - \frac{1}{1/\lambda\lambda_z \times \lambda R} \right] \left(\lambda + \frac{\partial \lambda}{\partial R} R \right) dR$$

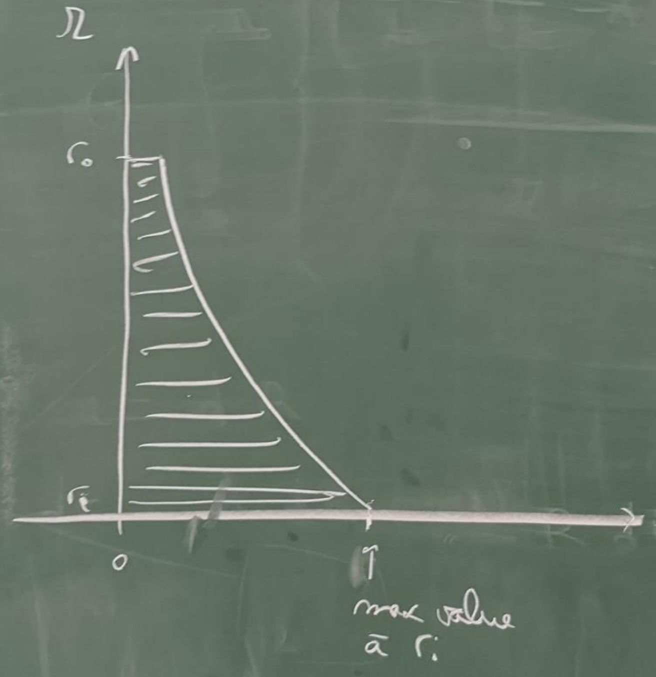
$$p = G \int_{R_i}^{R_o} \left(\frac{1}{R} - \frac{\lambda_z}{R} \right) \left(\lambda + \lambda' R \right) dR$$

$$\lambda = \sqrt{\frac{1}{\lambda^2} - \left(\frac{1}{\lambda^2} - \lambda_i^2\right)\frac{R_i^2}{R^2}}$$

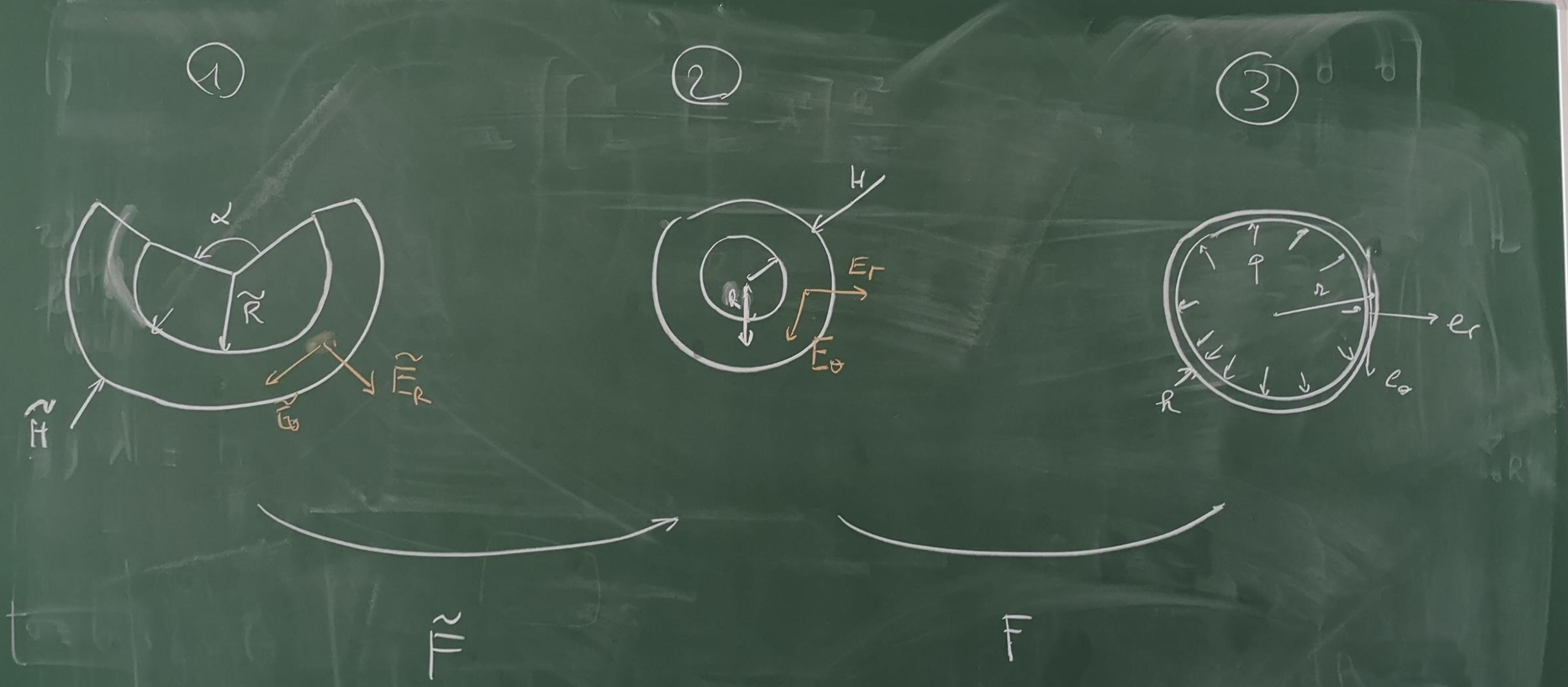
$$\lambda' = \left(\frac{1}{\lambda^2} - \lambda_i^2\right)\frac{R_i^2}{R^3} \Bigg/ \sqrt{\frac{1}{\lambda^2} - \left(\frac{1}{\lambda^2} - \lambda_i^2\right)\frac{R_i^2}{R^2}}$$

r
r_0
r_1
$\sigma_{\theta\theta} - \sigma_{rr}$





$$\sigma_{\theta\theta} - \sigma_{rr} = G\left(\lambda^2 - \frac{1}{\lambda_z^2\lambda^2}\right)$$



$$F^{tot} = \tilde{F}\, F$$

$$\tilde{F} = \begin{bmatrix} \frac{1}{\tilde{\lambda}} & 0 & 0 \\ 0 & \tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{1}{\lambda\lambda_2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\sigma = G \underbrace{F^T F}_{F^{tot}} + c I$$

$$(2\pi - \alpha) \tilde{R} \tilde{H} = 2\pi R H$$

$$\tilde{\lambda} = \frac{R}{\tilde{R}} = \frac{2\pi - \alpha}{2\pi} \qquad \frac{\tilde{H}}{H} = \frac{2\pi - \alpha}{2\pi} \frac{1}{\tilde{\lambda}}$$

$$\tilde{\lambda}^2 = \frac{2\pi - \alpha}{2\pi}$$