Journal of Physics: Conference Series **135** (2008) 012045 doi:10.1088/1742-6596/135/1/012045

Comparison between the basic least squares and the Bayesian approach for elastic constants identification

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Abstract. The basic formulation of the least squares method, based on the L_2 norm of the misfit, is still widely used today for identifying elastic material properties from experimental data. An alternative statistical approach is the Bayesian method. We seek here situations with significant difference between the material properties found by the two methods. For a simple three bar truss example we illustrate three such situations in which the Bayesian approach leads to more accurate results: different magnitude of the measurements, different uncertainty in the measurements and correlation among measurements. When all three effects add up, the Bayesian approach can have a large advantage. We then compared the two methods for identification of elastic constants from plate vibration natural frequencies.

1. Introduction

Identifying parameters of a model using experimental data has been extensively studied, including for determining elastic material properties from strain or vibration measurements. In order to find the parameters that agree best with the experiments, the most widely used method is based on minimizing the least squares error between the experimental data and the model predictions. Even if many improvements have been brought to the method [1]-[3], the simplest formulation of the least squares method, based on minimizing the L_2 norm of the misfit [1], is still extensively used today. An alternative statistical method for parameter identification is based on Bayes' rule. The Bayesian approach is among the most general statistical frameworks since it can account for prior knowledge and also provides the uncertainty in the parameters identified. Isenberg proposed a Bayesian framework for parameter estimation in 1979 [4] and others applied this approach to frequency or modal identification, i.e. identifying material properties from vibration test data [5][6].

While both the least squares and the Bayesian methods have been used numerous times for parameter identification, we could not find any publications comparing the two on the same problem. The aim of this article is to compare the two, first on a simple example, then on a material property identification problem from vibration test data. We seek to identify situations where the Bayesian approach leads to significantly different results compared to the basic least squares method.

First, a three bar truss didactic example will be introduced for comparing the two methods. These will then be applied to the identification of elastic properties from natural frequencies of a plate.

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2. Material properties identification on a three bar truss example

2.1. Description of the three bar truss example

The first example is a simple material property identification problem on which the application of both approaches is straight forward, avoiding complexity which could cloud the comparison. The truss is under a horizontal force *p* and a vertical force *r* as shown in Figure 1. All three bars are assumed to have the same Young modulus *E* of 10 GPa, which is unknown and which we want to identify from strain measurements on two or three of the bars. The cross sectional areas of the bars are known exactly: A_A is the cross sectional area of bars A and C and A_B the cross sectional area of bar B.

The magnitudes of the forces *p* and *r* are uncertain with both normally distributed with a mean value of 10^4 N for *p* and 10^5 N for *r* and a standard deviation of 500 N for both. From static analysis we find the following relationships for the strains in the bars.

$$\varepsilon_A = \frac{1}{E}\left(\frac{r}{4A_B + A_A} + \frac{p}{\sqrt{3}A_A}\right) \quad (1)$$

$$\varepsilon_C = \frac{1}{E}\left(\frac{r}{4A_B + A_A} - \frac{p}{\sqrt{3}A_A}\right) \quad (2)$$

$$\varepsilon_B = \frac{1}{E}\frac{4r}{4A_B + A_A} \quad (3)$$

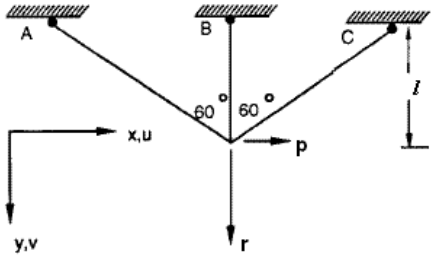

Figure 1. Three bar truss problem

2.2. The least square method

The simplest least squares formulation finds the parameters that minimize the sum of the squares of the errors between experimental observations and corresponding model predictions. Again note that more advanced formulations exist, but here we are interested in this simple yet still widely used formulation. For the three-bar truss, assuming we measure the strains in bars A and B, the least squares objective function that we seek to minimize can be written as:

$$J(E) = \left(\varepsilon_A^{nominal} - \varepsilon_A^{measure}\right)^2 + \left(\varepsilon_B^{nominal} - \varepsilon_B^{measure}\right)^2 \quad (4)$$

Note that even though the loads *p* and *r* are uncertain we have to provide a single nominal value for each. The most natural candidates are the means of the distributions of *p* and *r*. Note also that in this simple case it was possible to find the minimum analytically, without using numerical strategy.

2.3. The Bayesian method

The Bayesian approach is a statistical approach, giving not only a single value but a probability distribution. Compared to other statistical approaches to identification, such as maximum likelihood, the Bayesian approach is more general since it can incorporate prior knowledge over the parameters to be identified. It is based on the application of Bayes' rule which gives the probability *P(A/B)* of an event *A*, knowing the event *B* occurred as shown in Eq. 5 together with its extension to continuous probability distribution functions (PDF). Often *P(A)* is called the prior probability of A, $P^{prior}(A)$, to mark the distinction to the probability of *A* knowing *B*, also called probability of *A* posterior to *B*.

$$P(A/B) = \frac{P(B/A)\cdot P(A)}{P(B)}, \qquad f(A/B) = \frac{f(B/A)\cdot f^{prior}(A)}{\int f(B/A)\cdot f^{prior}(A)\cdot dA} = \frac{1}{K}\cdot f(B/A)\cdot f^{prior}(A) \quad (5)$$

Applying Eq. 5 to the three bar truss problem we can write the distribution of the Young modulus of the bars knowing that we measured $\varepsilon_A = \varepsilon_A^{mesure}$ and $\varepsilon_B = \varepsilon_B^{mesure}$ as shown in Eq. 6.

$$f\left(E \middle/ \left\{\varepsilon_A = \varepsilon_A^{mesure}, \varepsilon_B = \varepsilon_B^{mesure}\right\}\right) = \frac{1}{K} f\left(\left\{\varepsilon_A = \varepsilon_A^{mesure}, \varepsilon_B = \varepsilon_B^{mesure}\right\} \middle/ E\right) \cdot f^{prior}(E) \qquad (6)$$

The right hand side of this equation is composed, apart from the normalization constant *K*, of two quantities. The first is the likelihood function of *E* given the measurements and the other is the prior probability distribution of *E*. Here we assume that the prior knowledge is in the form of a normal distribution with mean value 9.5 GPa and standard deviation 1.5 GPa. This is a wide prior, centred relatively far away from the true value of 10 GPa to avoid biasing too much our comparison in favour of the Bayesian identification procedure. More details on the impact of the prior will be given in 4.5.

The other right hand side item in Eq. 6 is the likelihood function of *E* given the measurements $\varepsilon_A = \varepsilon_A^{mesure}$ and $\varepsilon_B = \varepsilon_B^{mesure}$. It measures the probability of getting the test result for a given true value of the modulus, and consequently, it provides an estimate of the likelihood of different modulus values given the test results. As we vary *E* successively from $-\infty$ to ∞, we calculate the joint probability distribution function of the strains for that *E* at the measured strain point $\{\varepsilon_A = \varepsilon_A^{mesure}, \varepsilon_B = \varepsilon_B^{mesure}\}$, that is $f\left(\left\{\varepsilon_A = \varepsilon_A^{mesure}, \varepsilon_B = \varepsilon_B^{mesure}\right\} \middle/ E\right)$. For a given *E* we have a PDF for the strains, due to the uncertainty in the loads *p* and *r*, which propagate to the strains.

The likelihood function for a given *E* is calculated using Monte Carlo simulation to generate 100,000 loads *p* and *r* and propagate them to strains using Eqs. 1-3. The fitted strain PDF is then taken at the point $\{\varepsilon_A = \varepsilon_A^{mesure}, \varepsilon_B = \varepsilon_B^{mesure}\}$. Equation 6 is evaluated in this way for a series of *E* values.

3. Least squares and Bayesian comparison for the three bar truss problem

3.1. The comparison method

The results of both the least squares and Bayesian approaches depend on the actual, but unknown, values of the loads *p* and *r* in an actual experiment. We compare the two methods in two different ways. First we use an extreme case where the actual values of *p* and *r* are two standard deviations away from their mean values: $p_{true}=p_m+2\sigma_p$ and $r_{true}=r_m-2\sigma_m$. Second we consider 1000 repetitions of the identification processes where the true values of *p* and *r* are obtained by Monte Carlo simulations from their distributions. This second case provides the average performance of each method.

For all the cases we compare the modulus obtained from the least squares approach to the most likely value from the Bayesian probability distribution. The differences between the two methods are likely to be influenced by three factors (i) differences in the magnitude of the measured strains; (ii) differences in the uncertainty of the measured strains; and (iii) correlation between measured strains.

3.2. Results for different-magnitude strains

To create a difference between the magnitude of the strain in bar A and the magnitude of the strain in bar B we used the numerical values given in Table 1. Note that while the strain in bar A is about three times higher than the one in bar B, we use the same relative uncertainty in the loads (2.5%), which propagates to about the same relative uncertainty in ε_A and ε_B.

Table 1. Numerical values for different-magnitude strains

	Input parameters						Measured strains*	
Parameter	A_A (m^2)	A_B (m^2)	p_m (N)	r_m (N)	σ_p (N)	σ_r (N)	ε_A (mε)	ε_B (mε)
Value	2.10^{-4}	1.10^{-2}	10^4	10^5	250	2500	3.26	0.945

* obtained from (1) and (3) with *E*= 10 GPa, $p=p_{mean}+2\sigma_p$ and $r=r_{mean}-2\sigma_m$

Table 2. Extreme case identification results for different-magnitude strain.

	From ε_A^{mesure} alone	From ε_B^{mesure} alone	Least squares	Bayesian
E (GPa)	9.59	10.52	9.67	9.97 (most likely value) 0.174 (standard deviation)

The results of the two identification procedures are presented in Table 2. We also provide the Young modulus that would be obtained with each of the measurements alone by inverting Eqs. 1 and 3 respectively. The relatively poor results by the least squares method are because least squares implicitly assigns more weight to a strain having high magnitude. Indeed looking at Eq. 4 we have $\varepsilon_A^{measure}$ which is about three times bigger than $\varepsilon_B^{measure}$, and so for $\sigma_A^{nominal}$ and $\sigma_B^{nominal}$. This means that for a same variation of E the residue resulting from bar A will be about 3^2 times bigger than the residue from bar B. So the least squares approach implicitly assigns about 9 times more weight to the measurement in bar A because it is 3 times higher in magnitude than the one in bar B. It seems logical then that the Young modulus identified by least squares is relatively close to the value found using $\varepsilon_A^{measure}$ alone. Note that which strain is assigned more weight depends on the derivatives of the strains with respect to E, which could change depending on the numerical values of the problem.

3.3. Results for different uncertainty in the strains

To create different uncertainty in the two strains we used the values given in Table 3. We chose 5% uncertainty in p and 0.5% in r. Note that in order to isolate this effect from the previous one, the obtained strains have about the same magnitude. The results of the two identification procedures are presented in Table 4 for the extreme case.

Table 3. Numerical values for different response uncertainty.

	Input parameters						Measured strains	
Parameter	A_A (m^2)	A_B (m^2)	p_m (N)	r_m (N)	σ_p (N)	σ_r (N)	ε_A (mε)	ε_B (mε)
Value	$7.85\ 10^{-4}$	$1\ 10^{-2}$	10^4	10^5	500	500	1.05	0.970

Table 4. Extreme case identification results for different response uncertainty.

	From ε_A^{mesure} alone	From ε_B^{mesure} alone	Least squares	Bayesian
E (GPa)	9.32	10.10	9.69	10.08 (most likely value) 0.058 (standard deviation)

Again on this extreme case the least squares approach is relatively far away from the true Young modulus of 10 GPa while the Bayesian approach is much closer. Since the two strains have about the same magnitude, least squares assigns about the same weight to each, so the identified E is at about the average between the E found with each measurement alone. However the two strains do not have the same uncertainty at all. By propagating the uncertainties on the loads to the strains we realize ε_A has about 7 times higher uncertainty than ε_B. This information is taken into account by the Bayesian method through the likelihood function, which can be seen as assigning more weight to the measurement having low uncertainty. This explains why the Bayesian identified modulus is much closer to the one found using $\varepsilon_B^{measure}$ alone.

3.4. Results for correlation among the responses

To show the effect of correlation we need three strain measurements with two strongly correlated but not correlated to the third. For this purpose we used the values in Table 1, simply adding from the strain in bar C, ε_C = -2.79 mε. The correlation between ε_A and ε_C is -0.985 while the correlation between the other two couples is 0.086, meaning that only ε_A and ε_C are highly correlated.

Table 6. Extreme case identification results for response correlation.

	From ε_A^{mesure}	From ε_B^{mesure}	From ε_C^{mesure}	Least squares	Bayesian
E (GPa)	9.59	10.52	9.43	9.58	9.96 (most likely value) 0.196 (standard deviation)

Comparison between Table 2 and Table 6 shows that the least squares method is more affected by adding a correlated measurement (+ 0.9 % error) than the Bayesian approach (+ 0.1% error). The explanation is that least square treats all three measurements as independent. Due to the small magnitude of ε_B^{mesure} it will be given a small weight, so least squares will mainly combine ε_A^{mesure} and ε_C^{mesure}. The Bayesian approach can be seen as averaging ε_A^{mesure} and ε_C^{mesure} first, then considering the average value as a single experiment it combines it with the uncorrelated one.

3.5. Results for all three effects together

In this last case we analyze what happens when all three effects act together, which is what may happen in a real situation. For this purpose we used the numerical values given in Table 7. We have different magnitude of the strains, different uncertainty in the loads and correlation among the strains: the correlation between ε_A and ε_C is -0.999, between ε_A and ε_B 0.014 and between ε_B and ε_C 0.002.

Table 7. Numerical values for all three effects.

	Input parameters						Measured strains		
Parameter	A_A (m^2)	A_B (m^2)	p_m (N)	r_m (N)	σ_p (N)	σ_r (N)	ε_A (mε)	ε_B (mε)	ε_C (mε)
Value	$2\ 10^{-4}$	$1\ 10^{-2}$	10^4	10^5	500	500	3.42	0.985	-2.93

Table 8. Extreme case identification results for all three effects simultaneously.

	From ε_A^{mesure}	From ε_B^{mesure}	From ε_C^{mesure}	Least squares	Bayesian
E (GPa)	9.16	10.10	9.00	9.14	10.08 (most likely value) 0.058 (standard deviation)

We can see in Table 8 that in this case the difference between the least squares and the Bayesian approach is exacerbated. All effects act together and against the least squares method. On the other hand the Bayesian method considers almost only ε_B , which has by far the lowest uncertainty, leading the Bayesian method to be much closer to the true Young modulus.

At this point we briefly discuss the influence of the prior distribution on the Bayesian identification results. The previous results were obtained for a normally distributed wide prior with mean value 9.5 GPa and a large standard deviation of 1.5 GPa. If we change the standard deviation of the prior to 0.75 GPa keeping the same mean we obtain on this extreme case a most likely value of 10.07 GPa (10.08 previously, see Table 8). If we change the mean of the prior distribution to 10.5 GPa the standard deviation remaining the same we obtain a most likely value of 10.09 GPa (compared to 10.08). So even though we changed the prior significantly it had very small effects on the Bayesian identification results. Of course this could have been different if we had a narrower, more accurate prior distribution available, in which case it could have positively affected the results.

3.6. Average performance

To complement the results obtained for the extreme case we repeat the previous procedure 1000 times, for random values of the loads obtained by Monte Carlo simulation. The quality of the methods will be measured mainly by the standard deviation of E as the loads are varied (see Table 9). For the different strain magnitude case for example the standard deviation of the Bayesian approach is about 17% lower than for least squares. This means that, on average, the E found by the Bayesian approach is about 17% closer to the true value. In all the cases the Bayesian approach systematically

outperforms least squares. The difference becomes even striking for the three effects combined, since, on average, the E found by the Bayesian approach will be almost 10 times closer to the true value.

On a final note, the previously analyzed situations can be handled appropriately by generalized least squares formulations as well, even though these might be less straight forward in obtaining the uncertainty in the identified parameters. The Bayesian approach is presented here as a natural alternative to the more complex least square formulations, which can appropriately handle the analyzed shortcomings of the basic least squares method. Also, even though not studied here, a main remaining advantage of the Bayesian approach over the generalized least squares formulations is the simplicity of taking into account prior information. In our examples we chose a wide prior though to concentrate on the comparison between the basic least squares and the statistical Bayesian approach.

Table 9. Average performance of the methods in the different cases.

		Mean of E (GPa)	Standard deviation of E (GPa)
Different magnitude	Least squares	10.01	0.200
	Bayesian	9.99	0.167
Different uncertainty	Least squares	10.00	0.178
	Bayesian	9.99	0.051
Correlation	Least squares	10.01	0.221
	Bayesian	9.99	0.167
All three together	Least squares	10.04	0.447
	Bayesian	9.99	0.050

4. Vibration identification problem

4.1. Description of the problem

In this section we explore how the two methods compare on a more realistic identification problem of elastic properties from the natural frequencies of a plate. Since we are only interested here in comparing the two methods, we simulate experiments from analytical expressions and error models. This also eases computational cost which would be a major issue if more complex models were used.

We consider a $[0, 90]_s$ simply supported composite laminate of dimensions a = 200 mm, b = 250 mm and of total thickness h = 3 mm. We assume the true elastic constants of the laminate are $E_x = E_y =$ 57.6 GPa, $G_{xy} = 4.26$ GPa and $v_{xy} = 0.05$. For simplicity, we identify only two properties, assuming that v_{xy} is known as well as $E_x = E_y$. This leaves E_x and G_{xy} to be identified.

The simulated experiment consists of measuring the first nine natural frequencies of the plate. We use the thin plate theory for the frequencies in terms of density ρ and rigidities D_{ij}. The assumed uncertainties in the thickness h, dimensions, and density are given in Table 9.

Table 9. Assumed uncertainties on the input parameters.

Parameter	Distribution
a	Normal(200mm , 0.500mm)
b	Normal(250mm , 0.500mm)
h	Normal(3mm , 10μm)
ρ	Normal(1535 kg/m^3, 7.67 kg/m^3)

$$f_{mn} = \frac{\pi}{2\sqrt{\rho h}} \sqrt{D_{11}\left(\frac{m}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m}{a}\right)^2\left(\frac{n}{b}\right)^2 + D_{22}\left(\frac{n}{b}\right)^4} \tag{7}$$

To simulate an experiment we generate random values for the input properties, and calculate the first nine natural frequencies using Eq. 7 to obtain the model response, denoted f^{resp}. However we do not measure this value exactly due to measuring noise or modeling error. We assume a uniform

measurement noise which gets higher for the higher natural frequencies because the higher vibration modes have smaller amplitudes than the lower modes. To this we add a systematic error, which is zero for the fundamental frequency and increases linearly. This error may account for the difference between thin and thick plate theory, since higher modes have shorter wavelength. The error model is then $f_{mn}^{measure} = f_{mn}^{resp} + u_{mn}$, where f^{resp} is the model response using Eq. 7 and the simulated values of *a, b, h* and ρ; *u* is a random variable uniformly distributed in the interval [a_{mn}, b_{mn}] where

$$a_{mn} = f_{11}^{resp}\left(a_{lb} + a_{ub}\frac{m+n-2}{m_{\max}+n_{\max}-2}\right) \qquad b_{mn} = f_{11}^{resp}\left(b_{lb} + b_{ub}\frac{m+n-2}{m_{\max}+n_{\max}-2}\right) \tag{8}$$

For nine frequencies m_{max}=n_{max}=3. We chose a_{lb}=*-2.5* 10^{-3}, a_{ub}=*-4* 10^{-2}, b_{lb}=*2.5* 10^{-3} and b_{ub}=*-2* 10^{-2}.

4.2. The identification methods

The least squares method minimizes the objective function shown in Eq. 9, where $f_{mn}^{resp}(E_x, G_{xy})$ is the response calculated using Eq. 7 using the mean values of *a, b, h* and ρ. We assume we know the average of the systematic error (a_{mn} + b_{mn})/2 for which we correct the experimental frequencies.

$$J(E_x, G_{xy}) = \sum_{m,n=1..3}\left(f_{mn}^{resp}(E_x, G_{xy}) - f_{mn}^{measure}\right)^2 \tag{9}$$

The Bayesian approach can be written as shown in Eq. 10.

$$f\left(E_x, G_{xy} \middle/ \left\{f_{11} = f_{11}^{mesure} \ldots f_{33} = f_{33}^{mesure}\right\}\right) = \frac{1}{K} f\left(\left\{f_{11} = f_{11}^{mesure} \ldots f_{33} = f_{33}^{mesure}\right\} \middle/ E_x, G_{xy}\right) \cdot f^{prior}(E_x, G_{xy}) \tag{10}$$

The main difference with the three bar truss formulation is that we work with the joint probability distribution of E_x and G_{xy} instead of the one dimensional distribution of *E*. The likelihood is calculated with 50,000 simulations with the uncertainties in *a, b, h* and ρ.

Unlike for the three bar truss we now also have error which we can take into account in the Bayesian approach. We assume that we know there is some numerical noise and that thin plate theory overpredicts the natural frequency but we assume we don't know the exact amount. We use the following error model for the Bayesian identification: $f_{mn}^{measure} = f_{mn}^{resp} + \tilde{u}_{mn}$ where f^{resp} is the model response using Eq. 7 and $\tilde{u}$ is a random variable uniformly distributed in the interval $\left[\tilde{a}_{mn}, \tilde{b}_{mn}\right]$ where $\tilde{a}_{mn}$ and $\tilde{b}_{mn}$ are obtained using Eq. 8 with a_{lb}=*-5* 10^{-3}, a_{ub}=*-5* 10^{-2}, b_{lb}=*5* 10^{-3} and b_{ub}=*-1* 10^{-2}. These error bounds are significantly wider than the ones of the actual error model, reflecting the fact that we only have vague knowledge of the error model.

The prior distribution was assumed to be uncorrelated bi-normal with mean 57 GPa and standard deviation 10 GPa for E_x and 4.2 GPa and 1.5 GPa respectively for G_{xy}. This is again a wide distribution to avoid that the prior gives the Bayesian method an unfair advantage.

4.3. Results comparison

To illustrate some benefits of the Bayesian method we present first the results for one particular simulation where we randomly simulated a single experiment (simulated as explained previously).

On this case the least squares procedure identified E_x = 57.9 GPa and G_{xy} = 4.61 GPa. Recall the true values are E_x = 57.6 GPa and G_{xy} = 4.26 GPa . The Bayesian approach obtained the distribution shown in Figure 2. The maximum of the distribution is in E_x = 57.9 GPa , G_{xy} = 4.30 GPa . Both approaches found an E_x which is very close to the true value (0.5%). However G_{xy} found by the least squares is 8.2% off the true value while the one found by the Bayesian method is only 0.9% off.

Furthermore the Bayesian approach provides additional information in form of the standard deviation of the distribution. Note that the distribution along G_{xy} is much wider than the one along E_x (note different scales in Figure 2). This means that the confidence in the most likely value of G_{xy} is

much poorer than in the one of E_x. This reflects the well known fact that G_{xy} is harder to identify accurately than E_x from a vibration test.

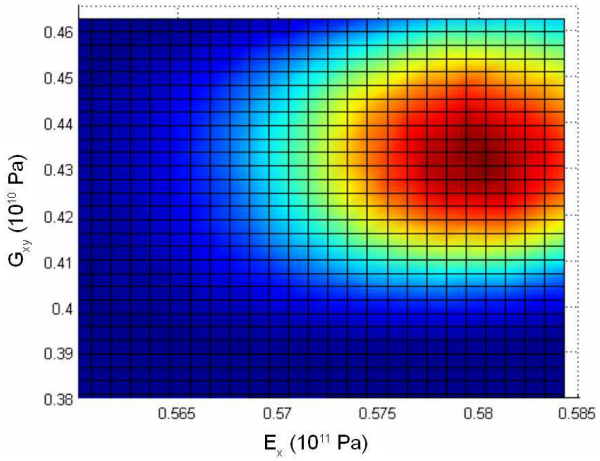

Figure 2. Posterior (E_x , G_{xy}) distribution found with the Bayesian approach

The average performance results over 100 repetitions are given in Table 11. The two methods are comparable for E_x but the Bayesian approach is about 1.9 times more accurate for G_{xy} , because the least squares objective function is very sensitive to E_x but much less to G_{xy}.

Table 11. Average performance for the plate vibration problem with 100 repetitions.

	Mean value (GPa)	Standard deviation (GPa)
Least squares	E_x = 57.5 ; G_{xy} = 4.26	For E_x : 0.65 (1.13%) ; for G_{xy}: 0.15 (3.63%)
Bayesian	E_x = 57.5 ; G_{xy} = 4.26	For E_x : 0.50 (0.88%) ; for G_{xy}: 0.083 (1.96%)

5. Concluding remarks

We compared for two different problems two approaches to parameter identification: a basic least squares approach and a Bayesian approach. Using a three bar truss didactic example we identified the following conditions under which the basic least squares method is systematically outperformed by the Bayesian method: different magnitude of response components, different uncertainty in the measurements and correlation among the measurements. The amplitude of the difference between the two approaches depends on the specific problem but on the truss problem we illustrated that it can reach a factor of ten when all the effects act combined.

We then considered the identification of elastic constants from natural frequencies of a plate. Using simulated experiments affected by uncertainty in input parameters, measurement noise and model error, we compared the two identification approaches. We found that the Bayesian approach presented an advantage in particular for identifying parameters to which the response is relatively insensitive.

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