Improving Max-Sum through Decimation
to Solve Cyclic Distributed Constraint Optimization Problems

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What problems are we dealing with?

Problems represented as factor graphs
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Distributed Constraint Optimization Problems (DCOPs)
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Problems represented as factor graphs

Distributed Constraint Optimization Problems (DCOPs)

One possible and often efficient solution method to find $\max_{\mathcal{X}} \sum_{m=1}^{M} f_m(\mathcal{X}_m)$: Max-Sum [Farinelli et al., 2008]
What's Max-Sum?
Belief-Propagation-based message passing algorithm

Each variable/factor sends messages:

$$q_{n \to m}(x_n) = \alpha_{nm} + \sum_{m' \in \mathcal{V}(n) \setminus m} r_{m' \to n}(x_n)$$  \hspace{1cm} (1)$$

$$r_{m \to n}(x_n) = \max_{\mathcal{X}_m \setminus n} \left( f_m(\mathcal{X}_m) \sum_{n' \in \mathcal{F}(m) \setminus n} q_{n' \to m}(x_{n'}) \right)$$  \hspace{1cm} (2)$$

and computes a marginal function:

$$z_n(x_n) = \max_{\mathcal{X}_m \setminus n} \sum_{m=1}^{M} f_m(\mathcal{X}_m)$$  \hspace{1cm} (3)$$
What’s Max-Sum? (cont.)

In the end, each variable acquires belief about its influence on the overall objective → decoding to get the solution \((\text{argmax } z_n(x_n))\)
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What’s the problem with Max-Sum (MS)?

- On tree-shaped FGs: MS proven to converge to optimal solutions
- In more general cyclic settings:
  - May converge to non optimal solutions
  - May not converge at all

Here, convergence means the marginal functions do not change for a while...
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Several approaches to handle loops in MS

- Bounded MS [Rogers et al., 2011]
- Max-Sum_AD_VP [Zivan et al., 2017]
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But let’s also have a look at...

- Decimation [Montanari et al., 2007], coming from statistical physics to solve $k$-satisfiability loopy problems
What’s decimation?

**Simple principle** = alternating belief-propagation (BP) and assignment of values to some variables depending on their marginal value, until all variables have been assigned a value

Example (implementing *Montanari et al.*, 2007)
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![Diagram of variable interactions with marginal plots](image-url)
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![Diagram of a factor graph](image-url)
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**Example (implementing [MONTANARI et al., 2007])**

BP → choosing a variable randomly → sampling the value of the variable depending on its marginal
→ BP → ...

\[
x_1 \quad f_1 \quad x_2 \quad f_2 \quad x_3 \quad f_3
\]
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![Diagram](figure.png)
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**Example (implementing [Montanari et al., 2007])**
BP → choosing a variable randomly → sampling the value of the variable depending on its marginal → BP → ...

```
x_1

\[
\begin{array}{c}
  \text{f}_1 \\
  \text{x}_2 \\
  \text{f}_3 \\
  \text{x}_3
\end{array}
\]
```

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**Example (implementing [MONTANARI et al., 2007])**

BP $\rightarrow$ choosing a variable randomly $\rightarrow$ sampling the value of the variable depending on its marginal $\rightarrow$ BP $\rightarrow$ ...

\[
\begin{align*}
x_1 & \quad f_2 \quad x_3 \\
f_1 & \quad x_2 \quad f_3
\end{align*}
\]
Let’s generalize and try to use decimation in Max-Sum

To install decimation in a BP-based solution method, we need to identify

1. when decimation should be triggered
   ▶ each time step, each $n$ time steps, once a loop is detected, ...

2. the subset of variables to decimate
   ▶ one variable randomly, one variable with some properties, several variables, ...

3. the values to assign to decimated variable(s)
   ▶ sampling on marginal values, most determined value, ...

We call a decimation policy any combination of (1), (2) and (3)

Our idea here is to apply different decimation policies to Max-Sum
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→ A generic decimation framework for Max-Sum, a.k.a DeciMaxSum
DeciMaxSum as an algorithm

**Data:** A factor graph $FG = \langle X, C, E \rangle$, a decimation policy $\pi = \langle \Theta, \Phi, \Upsilon, \Lambda \rangle$

**Result:** A feasible assignment $X^*$

- initialize BP messages $U \leftarrow \emptyset$
- while $U \neq X$ do
  - run BP until decimation triggers, i.e. $\Theta(FG^t) = 1$
  - choose variables to decimate, $X' = \{x_i \in \Phi(FG^t) \mid \Upsilon(x_i, FG^t)\}$
    - for $x_i \in X'$ do
      - $x_i \leftarrow \Lambda(x_i, FG^t)$
      - $U \leftarrow U \cup \{x_i\}$
      - simplify $FG^t$  
        // remove variables, slice factors

- return $X^*$ by decoding $U$
Implementing [MONTANARI et al., 2007] in DECI\textsc{MaxSum}

1. decimate once BP converges (or halt after some time limit)
2. choose on random variable within the whole set of non decimated variables
3. sampling the value wrt the marginal values
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1. decimate once BP converges (or halt after some time limit)
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3. sampling the value wrt the marginal values
Implementing [MOOIJ, 2010] in DECIMAXSUM

1. decimate once BP converges (or halt after some time limit)
2. choose the most determined variable, i.e. the lowest entropy $H$ on marginal values, within the whole set of non decimated variables

$$H(z_k(x_k)) = - \sum_{d \in \mathcal{D}_k} z_k(x_k)(d) \log(z_k(x_k)(d))$$

3. choose the value with highest marginal value ($\arg\max_{d \in \mathcal{D}_i} z_i(x_i)(d)$)
Implementing [Mooij, 2010] in DeciMAXSum

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And many more combinations...

- DECIMAXSUM (2-periodic, min-entropy, deterministic)
- DECIMAXSUM (3-periodic, min-entropy, deterministic)
- DECIMAXSUM (4-periodic, min-entropy, deterministic)
- DECIMAXSUM (5-periodic, min-entropy, deterministic)
- DECIMAXSUM (10-periodic, min-entropy, deterministic)
- DECIMAXSUM (20-periodic, min-entropy, deterministic)
- DECIMAXSUM (100-periodic, min-entropy, deterministic)
- DECIMAXSUM (10-periodic, random, sampling)
- DECIMAXSUM (100-periodic, random, sampling)
- DECIMAXSUM (periodic, min-entropy, deterministic)
- DECIMAXSUM (convergence, min-entropy, deterministic)

- MaxSum
- Montanari-Decimation
- Mooij-Decimation
- ...

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Benchmarking on a very cyclic problem: the Ising model

- toroidal grid structure
- boolean variables $x_i$'s
- unary costs $r_i$'s
- binary constraints $r_{ij}$'s
Quality of solutions

Final cost comparison on Ising problems

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Quality of solutions (cont.)

Final cost comparison on Ising problems

- DeciMaxSum (2-periodic, min-entropy, deterministic)
- MaxSum
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- Montanari-Decimation
- Mooij-Decimation
Communication load

Communication costs on Ising problems

DeciMaxSum (10-periodic, min-entropy, deterministic)
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Communication load (cont.)

Communication costs on Ising problems

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Conclusions

To sum up

■ We have proposed a generic framework to integrate decimation mechanism into MaxSum/BP algorithms

■ On very cyclic problems (Ising model), fast decimation based on marginal function entropy and deterministic value assignment showed very good quality solutions, with many less messages

■ Decimation $\equiv$ decoding at runtime?

Many ways to go

■ Many more policies are possible
  ▶ ex: decimation once a loop is detected
  ▶ ex: alternating deterministic and non deterministic value assignment

■ How does decimation behave on less cyclic but less regular problems?

■ How does decimation behave on non boolean settings?
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