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Analysis of a class of k-dimensional merge procedures, with an application to $2d$ Delaunay triangulation in expected linear time after two-directional sorting

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Extended abstract

Abstract *This paper exploits the notion of "unfinished sites" in the average-case analysis of k-dimensional divide-and-conquer algorithms.*

This general result is then applied to the 2D case, and it is shown that the divide-and-conquer construction of the Delaunay triangulation of a set of planar points quasi-uniformly distributed in a square may be done in expected linear time after a two-directional preprocessing sort. This method is readily implemented, and, as shown by the easily reproduced results provided, it is one of the fastest worst-case optimal methods ever suggested to construct real-scale Delaunay triangulations in the plane.

1 Introduction

The first worst-case optimal, $\Theta(n \log n)$, method to construct the Delaunay triangulation in the plane was published in 1980 by Lee and Schachter ([8]). Their method – which will be referred to in the text as "the standard algorithm" – was based on the famous divide-and-conquer paradigm. In 1985, Guibas and Stolfi generalized this method ([6]). Meanwhile, in 1984, A. Maus ([9]) published the first expected linear-time method for constructing the Delaunay triangulation of a site of planar points, using a grid to speed up the location of the closest empty-circle neighbour for any given Delaunay edge.

In 1987, R. Dwyer published an $O(n \log\log n)$ method ([5]) for the same problem assuming uniform distribution, using a regular grid. In their 1988 paper, [7], J. Katajainen and M. Koppinen introduced a variant of the standard algorithm, based on a regular grid. Their average-case analysis exploited the notion of unfinished sites in a rectangular domain. Finally, R. Dwyer published the first k-dimensional method with expected linear behaviour in 1991 ([4]).

This paper addresses the analysis of k-dimensional divide-and-conquer methods that make use of the notion of unfinished site and of (balanced) kd-tree based partitioning. Some preliminary definitions and mathematical results are given in Section 2. Section 3 then presents the main result for dimension k, that is exploited, in Section 4, to obtain the construction of the Delaunay triangulation of a set of planar points quasi-uniformly distributed in a square, in expected linear time, after a two-directional sort preprocessing step.

2 Definitions and mathematical preliminaries

We first need to prepare the ground for the general analysis of Section 3 by presenting its framework and by giving some preliminary definitions or results that will be required afterwards.

Although the analysis has a wider scope, let us, for the time being, concentrate on the k-dimensional Delaunay triangulation, noted $DT(S)$, of a set S of *sites*. We shall use the term "triangulation" for k-triangulation ($k \geq 2$) throughout this paper.

Assume the sites to be quasi-uniformly distributed on a unit cube $\mathcal{U}_k$. This implies the existence of two strictly positive real constants $c_1 \leq c_2$, and of a probability density f such that,

$$\forall X = (x_1, x_2, \ldots, x_k) \in \mathcal{U}_k, c_1 \leq f(X) \leq c_2,$$

and f is null outside the cube. Thus, the probability for one given site to lie in domain $\mathcal{D}$ is $\int_{\mathcal{D}} f$.

2.1 Unfinished sites

Let us generalize some definitions and statements from [7]:

Definition 2.1 *Let $T(S_1)$ and $T(S_2)$ be the triangulations of sets S_1 and S_2. We shall write $T(S_1) <_\Delta T(S_2)$ whenever $S_1 \subseteq S_2$ and $T(S_1)$ contains each edge in $T(S_2)$ the endpoints of which belong to S_1.*

Definition 2.2 *Let $T(S_1) <_\Delta T(S_2)$ be two triangulations and $s \in S_1$. We say that site s is* finished *in*

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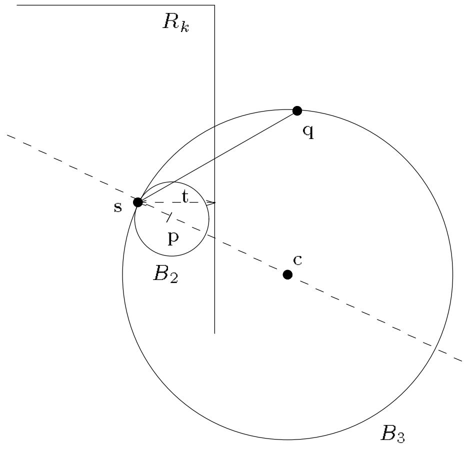

Figure 1: *Lemma 2.1.*

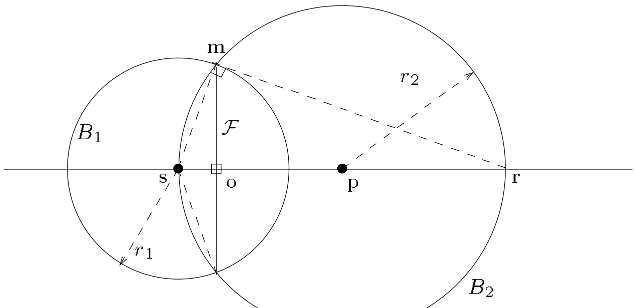

Figure 2: *Lemma 2.2.*

$T(S_1)$ wrt $T(S_2)$ if the set of edges adjacent to s in $T(S_1)$ and $T(S_2)$ coincide; otherwise, s is said to be unfinished *(in $T(S_1)$ wrt $T(S_2)$).*

The first definition induces a partial order in the set of triangulations. Notice that $T(S_1) <_\Delta T(S_2)$ does not imply that all edges in $T(S_1)$ belong to $T(S_2)$. Also note that $DT(S_1) <_\Delta DT(S_2)$ is equivalent to $S_1 \subseteq S_2$ if S_1 has a unique Delaunay triangulation.

The two following results are direct consequences of Definition 2.2:

Proposition 2.1 *Let $T(S_1) <_\Delta T(S_2)$ be two triangulations. A site $s \in S_1$ is finished in $T(S_1)$ wrt $T(S_2)$ if and only if S_1 contains the endpoints of all edges adjacent to s in $T(S_2)$.*

2.2 k-partition around unfinished sites

Lemma 2.1 *If s is an unfinished site in k-rectangle $\mathcal{R}_k$, at distance t from the boundary of $\mathcal{R}_k$, then there is a k-ball $\mathcal{B}_2$ with radius $\frac{t}{2}$, centered at distance $\frac{t}{2}$ from s, with no sites in its interior.*

Proof: Referring to Figure 1, if s is an unfinished site in k-rectangle $\mathcal{R}_k$, there is a site q belonging to $\mathcal{U}_k \setminus \mathcal{R}_k$ such that (s, q) is a Delaunay edge in $\mathcal{U}_k$. Since q lies outside k-rectangle $\mathcal{R}_k$, the length of (s, q) is greater than t. And since (s, q) is a Delaunay edge, we may find a k-ball $\mathcal{B}_3$ with no sites in its interior, and s and q on its boundary. The diameter of this k-ball is thus greater than t. Let c be the center of $\mathcal{B}_3$, and $\mathcal{B}_2$ the k-ball with center lying on segment $[sc]$ at distance $\frac{t}{2}$ from s. $\mathcal{B}_2$ is contained in $\mathcal{B}_3$, hence $\mathcal{B}_2$ has no sites in its interior.

Lemma 2.2 *Let s be a site, and $\mathcal{B}_1$ the ball centered in s, with radius r_1. Let $\mathcal{B}_2$ be the ball with radius $r_2 = \frac{t}{2}$, and center p at distance $\frac{t}{2}$ from s. Let $(s, \vec{\imath}_1, \ldots, \vec{\imath}_k)$ be an orthonormal system of coordinates, with $\vec{\imath}_1$ collinear to $\overrightarrow{sp}$ and with same direction. Let $\mathcal{R}o$ be the portion of 'rectangular' cone with apex s, with symmetry axis $\vec{\imath}_1$, included in $\mathcal{B}_1$, and with equation in a generalized spherical coordinate system $(\rho, \alpha_1, \ldots, \alpha_{k-1})$:*

$$(\mathcal{R}o)\left\{\begin{array}{cccc} \rho & < & r_1 & \\ -\frac{\pi}{4} & < & \alpha_1 & < & \frac{\pi}{4} \\ \vdots & & \vdots & & \vdots \\ -\frac{\pi}{4} & < & \alpha_{k-1} & < & \frac{\pi}{4} \end{array}\right.$$

then $$\mathcal{R}o \subset \mathcal{B}_2 \Leftrightarrow r_1 \leq \frac{t}{2^{\frac{k-1}{2}}}$$

Proof: Let us first show that $\mathcal{R}o \subset \mathcal{B}_2 \Rightarrow r_1 \leq \frac{t}{2^{\frac{k-1}{2}}}$.

Knowing that $\mathcal{R}o \subset \mathcal{B}_2$ implies that, for all $\alpha_i = \pm\frac{\pi}{4}$, $m(r_1, \alpha_1, \ldots \alpha_{k-1})$ belong to $\mathcal{B}_2$.

Now, consider the limiting case where points m are situated on the common boundary of balls $\mathcal{B}_1$ and $\mathcal{B}_2$ (Figure 2). We have : $so = \dfrac{r_1}{(\sqrt{2})^{k-1}}$

Since triangles $\triangle som$ and $\triangle smr$ are similar:

$$\frac{so}{sm} = \frac{sm}{sr} \Rightarrow \frac{1}{(\sqrt{2})^{k-1}} = \frac{r_1}{t} \Rightarrow r_1 = \frac{t}{2^{\frac{k-1}{2}}}$$

whence: $\mathcal{R}o \subset \mathcal{B}_2 \Rightarrow r_1 \leq \dfrac{t}{2^{\frac{k-1}{2}}}$

The reciprocal is straightforward.

Corollary 2.1 *Consider te $(k-1)$-ball b_{k-1} circumscribed to the vertices of the rectangular section $\mathcal{F}$ of $(\mathcal{R}o)$. Let $(\mathcal{C}o)$ be the cone with section b_{k-1} and apex s. Then,*

$$\mathcal{R}o \subset \mathcal{B}_2 \Leftrightarrow \mathcal{C}o \subset \mathcal{B}_2$$

Proof: Suppose that $\mathcal{R}o \subset \mathcal{B}_2$. Ball b_{k-1} circumscribed to $\mathcal{F}$ has radius om and center o (see Figure 2).

It is therefore included in the intersection of $\mathcal{B}_1$ and $\mathcal{B}_2$. Hence, cone $\mathcal{C}o$ is included in $\mathcal{B}_2$. The reciprocal is straigtforward, since $\mathcal{R}o \subset \mathcal{C}o$.

Remark: If $\mathcal{R}o$ is rotated around its symmetry axis, it is still included in $\mathcal{C}o$, and hence in $\mathcal{B}_2$.

Corollary 2.2 *If s is an unfinished site inside k-rectangle $\mathcal{R}_k$, and at distance t from its boundary, then we may find a k-partition of the neighbourhood of s such that at least one of the partitioning k-cells – each with volume $\dfrac{\pi^{\frac{k}{2}} t^k}{k 2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}$ – has no sites in its interior.*

Proof: If s is unfinished, Lemmas 2.1, 2.2 and Corollary 2.1 imply that we may find one open cone wedge $\mathcal{C}o_s$, with vertex s and no sites in its interior. However, $\mathcal{C}o_s$ depends on the position of the Delaunay edge originating from s.

We shall then construct a fixed k-partition of ball $\mathcal{B}_1$ and show that, if s is unfinished, at least one of the partitioning k-cells of $\mathcal{B}_1$ has no sites in its interior.

Let $(s, \vec{\imath}_1, \ldots, \vec{\imath}_k)$ be an orthonormal system of co-ordinates. Chose in $\mathcal{C}o_s$ a 'rectangular' cone $\mathcal{R}o_s$ such that the $(d-2)$-faces of its section are orthogonal to the axes of the co-ordinate system. Using the spherical co-ordinates $(\rho, \alpha_1, \ldots, \alpha_{k-1})$, its equation, in the above co-ordinate system, will be of the form:

$$(\mathcal{R}o_s) \begin{cases} \rho < r_1 & \\ a_1 < \alpha_1 < a_1 + \frac{\pi}{2} \\ \vdots \quad \vdots \quad \vdots \\ a_{k-1} < \alpha_{k-1} < a_{k-1} + \frac{\pi}{2} \end{cases}$$

Notice that, if the a_i are judiciously chosen, it is possible to build a partition of ball $\mathcal{B}_1$ with 'rectangular' cones $\mathcal{R}o$.

The number of elements of the paving is equal to the number of faces of an hypercube (a cube is a regular polyedron inscribable in a ball, in all dimensions [2]). The faces of this hypercube differ little from the section of the $(\mathcal{R}o)$'s. (but are indeed different, except for $k = 2$)

Let us now consider, more precisely, a paving of ball $\mathcal{B}_1$ with 'rectangular' cones $\mathcal{R}o$ with symmetry axes equal to the axes of the co-ordinate system. Imagine that we divide the section of $(\mathcal{R}o)$'s along all hyperplanes orthogonal to the axes of the referential (except the one orthogonal to its symmetry axis), and containing the center of symmetry of the section of $(\mathcal{R}o)$'s. This partition yields 2^{k-1} "cells", each of which being associated to a cone wedge (ro) with vertex s, limited by $\mathcal{B}_1$.

$$(ro) \begin{cases} \rho < r_1 & \\ l_1 \frac{\pi}{4} < \alpha_1 < (l_1+1)\frac{\pi}{4} \\ \vdots \quad \vdots \quad \vdots \\ l_{k-1} \frac{\pi}{4} < \alpha_{k-1} < (l_{k-1}+1)\frac{\pi}{4} \end{cases}$$

We have found a *fixed* and finer partition of ball $\mathcal{B}_1$ by $2k \times 2^{k-1} = k2^k$ cone wedges, each with the same volume, for symmetry reasons.

Consequently, $(\mathcal{R}o_s)$ – and hence $(\mathcal{C}o_s)$–, will necessarily contain at least one partition cell (ro_j) void of sites, whatever the direction of its axis (*e.g.* set $l_i = \lceil \frac{4a_i}{\pi} \rceil$).

All cone wedges (ro) have the same volume, which is thus equal to the volume of the k-ball centered in s with radius $\frac{t}{2^{\frac{k-1}{2}}}$, divided by $(k2^k)$.

$$\mathcal{V}(ro) = \frac{\pi^{\frac{k}{2}} \left(\frac{t}{2^{\frac{k-1}{2}}}\right)^k}{\Gamma\left(\frac{k}{2}+1\right) k 2^k} = \frac{\pi^{\frac{k}{2}} t^k}{k 2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}$$

(where Γ is the second-order eulerian function).

2.3 Probability for a site to be unfinished

Lemma 2.3 *Assuming the probability density f to be quasi-uniform with bounds c_1 and c_2, consider rectangle $\mathcal{R}_k \subseteq \mathcal{U}_k$ in $DT(S \cap \mathcal{R}_k) <_\Delta DT(S)$. If $s \in S \cap \mathcal{R}_k$ is a site at distance t from the boundary of $\mathcal{R}_k$, then the probability $p(\mathcal{C}_1)$ for s to be unfinished in $DT(S \cap \mathcal{R}_k)$ wrt $DT(S)$ is, at most:*

$$p(\mathcal{C}_1) \leq k2^k \left(1 - \frac{c_1 \pi^{\frac{k}{2}} t^k}{k 2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}\right)^{n-1}$$

Proof: $(\vec{\imath}_1, \ldots, \vec{\imath}_k)$ being the same orthonormal basis as before, $\mathcal{B}_1$, centered in s with radius $\frac{t}{2^{\frac{k-1}{2}}}$, is partitioned by 'rectangular' cone wedges (ro_j), $j \in [1, k2^k]$. Using Corollary 2.2, condition $\mathcal{C}_1$ – that s is unfinished in $DT(S \cap \mathcal{U}_k)$ *wrt* $DT(S)$ – implies condition $\mathcal{C}_2$ – that at least one of the open partitioning k-cells (ro_j) is void of sites. Hence,

$$\begin{aligned} P(\mathcal{C}_1) \leq P(\mathcal{C}_2) &\leq \sum_{j=1}^{k2^k} P(S \cap (co_j)) = \emptyset) \\ &= \sum_{j=1}^{k2^k} \left(1 - \int_{(co_j)} f\right)^{n-1} \\ &\leq k2^k \left(1 - \frac{c_1 \pi^{\frac{k}{2}} t^k}{k 2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}\right)^{n-1} \end{aligned}$$

using property $f(x, y) \geq c_1$.

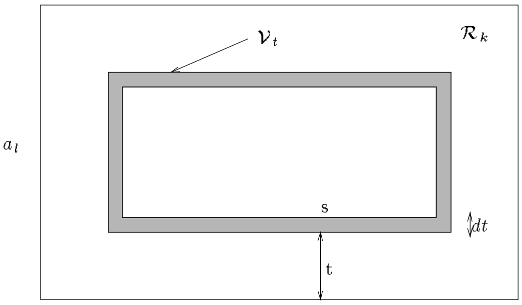

Figure 3: *Theorem 2.1*

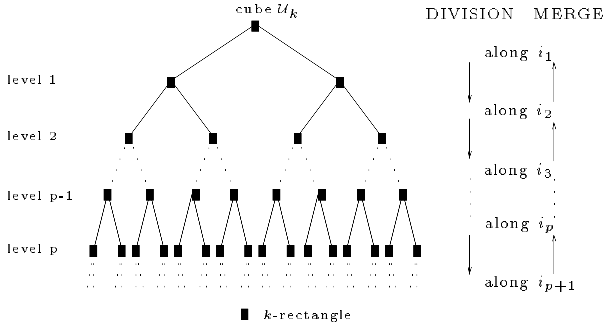

Figure 4: *Tree representation of divides and conquers according to a balanced kd-tree.*

2.4 Expected number of unfinished sites

Lemma 2.4 *For integers $n \geq 0$ and $k \geq 2$:*

$$\mathcal{I}_n^k = \int_0^1 \left(1 - x^k\right)^n dx = \prod_{i=1}^{n} \frac{ki}{ki+1} < \frac{1}{\sqrt[k]{n+1}}$$

(proof by induction).

Theorem 2.1 *In the setting of Lemma 2.3, if $\mathcal{S}_k$ is the surface of $\mathcal{R}_k$, and $E(\mathcal{R}_k)$ is the expected number of unfinished sites in $\mathcal{R}_k$, then:*

$$E(\mathcal{R}_k) \leq \frac{n}{\sqrt[k]{n}} \frac{c_2}{\sqrt[k]{c_1}} \mathcal{S}_k \frac{k2^{\frac{k(k+1)}{2}} \sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}}$$

Proof: Let $a_1, a_2, \ldots, a_k$ be the lengths of the sides of k-rectangle $\mathcal{R}_k$. Let a_l be the length of the smallest side of $\mathcal{R}_k$. Consider the elementary volume $\mathcal{V}_t$ comprising the points of $\mathcal{R}_k$ at a distance between t and $t+dt$ from the boundary of $\mathcal{R}_k$ (with $0 \leq t \leq \frac{a_l}{2}$) (Figure 3). Let us call $\mathcal{S}_t$ the exterior surface of $\mathcal{V}_t$:

$$\mathcal{S}_t = 2\sum_{j=1}^{k} \left(\prod_{i \neq j} (a_i - 2t) \right) \leq \mathcal{S}_k = 2\sum_{j=1}^{k} \left(\prod_{i \neq j} a_i \right)$$

$$\Rightarrow \mathcal{V}_t \leq \mathcal{S}_t \times dt \leq \mathcal{S}_k \times dt$$

The probability for a given site to belong to $\mathcal{V}_t$ is:

$$\int_{\mathcal{V}_t} f \leq c_2 \times \mathcal{V}_t \leq c_2 \times \mathcal{S}_k \times dt$$

Using Lemma 2.3, we have the following estimation:

$$\begin{aligned} E(\mathcal{R}_k) &\leq n\int_0^{\frac{a_l}{2}} k2^k \left(1 - \frac{c_1 \pi^{\frac{k}{2}} t^k}{k2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}\right)^{n-1} c_2 \mathcal{S}_k \, dt \\ &= n\frac{c_2}{\sqrt[k]{c_1}} \mathcal{S}_k \frac{k2^{\frac{k(k+1)}{2}} \sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}} \\ &\quad \times \int_0^{\frac{a_l\sqrt{\pi}}{2^{\frac{k(k+1)}{2}}} \sqrt[k]{\frac{c_1}{k\Gamma\left(\frac{k}{2}+1\right)}}} \left(1 - x^k\right)^{n-1} dx \end{aligned}$$

setting $x = t \sqrt[k]{\frac{c_1 \pi^{k/2}}{k2^{\frac{k(k+1)}{2}} \Gamma\left(\frac{k}{2}+1\right)}}$.

Since $a_l \leq 1$ et $c_1 \leq 1$, we may write:

$$\begin{aligned} E(\mathcal{R}_k) &\leq n\frac{c_2}{\sqrt[k]{c_1}} \mathcal{S}_k \frac{k2^{\frac{k(k+1)}{2}} \sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}} \\ &\quad \times \int_0^1 \left(1 - x^k\right)^{n-1} dx \\ &\leq \frac{n}{\sqrt[k]{n}} \frac{c_2}{\sqrt[k]{c_1}} \mathcal{S}_k \frac{k2^{\frac{k(k+1)}{2}} \sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}} \end{aligned}$$

using the upper bound on the integral from Lemma 2.4. Hence, $E(\mathcal{R}_k) = O(\frac{n}{\sqrt[k]{n}} \times \frac{c_2}{\sqrt[k]{c_1}} \times \mathcal{S}_k)$.

Remark: This bound is better than that obtained by Katajainen and Koppinen in the plane [7].

3 Average-case analysis of a class of k-dimensional merge procedures using the notion of unfinished sites

In this section, we demonstrate how the previous result may be exploited to analyze certain classes of multi-dimensional *divide-and-conquer*-type algorithms. Please refer to figures 4 and 5.

Theorem 3.1 *Let S be a set of n sites distributed in a k-dimensional unit hypercube $\mathcal{U}_k$, according to a quasi-uniform density probability f, with bounds c_1 and c_2 ($c_1 \leq c_2$). Consider the following algorithmic scheme:*

Divide step: *Divide $\mathcal{U}_k$ until reaching cells, each containing one single site, using a balanced kd-tree ([1]);*

Merge step: *Then re-construct $\mathcal{U}_k$ through successive merges in reverse order (of the divisions).*

If merging two subsets takes times proportional to the number of unfinished sites (wrt $\mathcal{U}_k$), then the whole merge *phase of the algorithm will be proportional to n.*

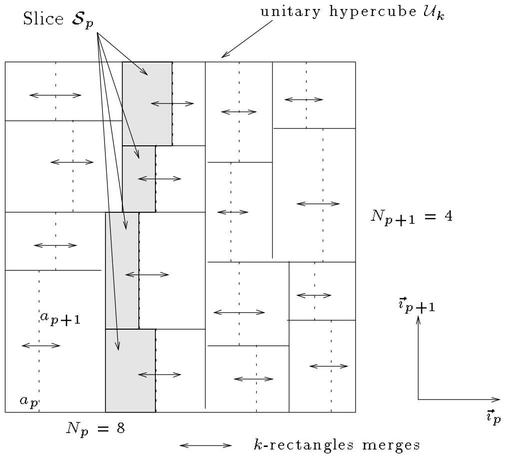

Figure 5: *Merges at level p: Slice S_p with all its (shaded) hyperrectangle constituents.*

Proof: Using Theorem 2.1, the running time of the *merge* phase will be proportional to

$$\frac{n}{\sqrt[k]{n}}\,\frac{c_2}{\sqrt[k]{c_1}}\,\frac{k2^{\frac{k(k+1)}{2}}\sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}}\sum_{k-rect.}\mathcal{S}$$

Hence, we must evaluate $\sigma = \sum_{k-rect.} \mathcal{S}$, this quantity representing the sum of the surfaces of all k-rectangles involved in the merges. We shall decompose σ into sub-sums σ_p, one for each merge level p.

Let $(\vec{\imath}_1, \ldots, \vec{\imath}_k)$ be an orthonormal basis, such that each axis is orthogonal to one hyperface of $\mathcal{U}_k$. The kd-tree produces a partition of $\mathcal{U}_k$, with n k-rectangles.

Definition 3.1 *Let us define* slice $\mathcal{S}_j$ *relative to direction* $\vec{\imath}_j$ *as one subset of the hyperrectangles partitioning* $\mathcal{U}_k$*, such that the orthogonal projection of* $\mathcal{S}_j$ *on either hyperface of* $\mathcal{U}_k$ *orthogonal to* $\vec{\imath}_j$ *constitutes a partition of this hyperface.*

During the construction of the kd-tree, the division of space – orthogonally to one direction – doubles the number of slices relative to this direction. Moreover, note that the whole rectangular partition for $\mathcal{U}_k$ may be itself partitioned into slices with the same direction, whatever this direction. Let us call N_j the number of slices relative to direction $\vec{\imath}_j$.

Consider merge-level p. Save for the last level – for which this value is only an upper bound for N_j – a simple induction argument yields:

$$N_j = 2^{\left\lceil \frac{p-j+1}{k} \right\rceil}.$$

Let us now evaluate σ_p by grouping k-rectangles into slices, and this, relatively to all directions. Since the surface of any hyperface of $\mathcal{U}_k$ is 1, the overall sum is equal to $2\sum_{j=1}^{k} N_j$, whence:

$$\frac{1}{2}\sigma_p \leq \sum_{j=1}^{k} 2^{\left\lceil \frac{p-j+1}{k} \right\rceil} \leq k2^{\left\lceil \frac{p}{k} \right\rceil}$$

Summing over all h merge-levels yields:

$$\frac{1}{2}\sigma = \frac{1}{2}\sum_{p=1}^{h}\sigma_p \leq \sum_{p=1}^{h} k2^{\left\lceil \frac{p}{k} \right\rceil} = k\sum_{p=1}^{h} 2^{\left\lceil \frac{p}{k} \right\rceil}$$

$$\leq k(k\sum_{w=1}^{\left\lceil \frac{h}{k} \right\rceil} 2^w) \leq k^2 2^{\left\lceil \frac{h}{k} \right\rceil + 1} \leq 4k^2 2^{\left\lceil \frac{h}{k} \right\rceil - 1}$$

Since the division scheme is supported by a balanced kd-tree, its own height h is such that $2^{h-1} < n \leq 2^h$, and hence:

$$\sqrt[k]{n} > 2^{\frac{h-1}{k}} \geq 2^{\left\lceil \frac{h}{k} \right\rceil - 1} \Rightarrow \sigma = \sum_{k-rect.} \mathcal{S} \leq 8k^2\sqrt[k]{n}$$

This implies that the average running time of the merge step is linear in the number of sites. It is, more precisely, proportional to:

$$n\frac{c_2}{\sqrt[k]{c_1}}\frac{k^3 2^{\frac{k(k+1)+6}{2}}\sqrt[k]{k\Gamma\left(\frac{k}{2}+1\right)}}{\sqrt{\pi}}$$

4 Application: $2d$ Delaunay triangulation in linear expected time after two-directional sorting

The previous result may be implemented in the construction of the Delaunay triangulation of a set of planar points, after a preprocessing two-directional sort. In general, the optimized algorithm proves to be 2 to 3 times faster than the standard D&C one [8]. For n distincts sites, the total number of edges created and of edges destroyed is close to $4n$ and n, respectively, and the number of edges of the initial triangulation still valid in the final one is close to $3n$.

Extensive tests were conducted on sets of up to seven millions sites, uniformly distributed in a square domain. Both the standard algorithm, and the standard-based optimized algorithm were tested on a Silicon Graphics Indy (200 MHz), and on a Convex C3 super-computer.

We have shown, in Figure 6, the results obtained on the C3. These results strongly corroborate the theory, and show the asymptotic $O(n \log n)$ and $O(n)$ behaviour of the standard and optimized versions, respectively (excluding preprocessing sorts). Ohya, Iri and Murota ([10]) have shown that *even* the average-case running time for the standard merge step is $\Omega(n \log n)$.

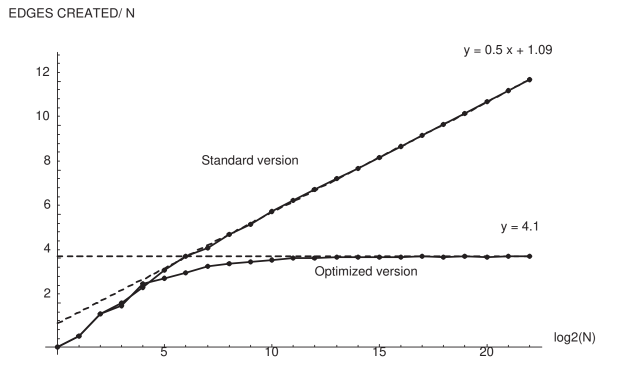

Figure 6: *Comparison of the standard (Lee and Schachter's) version, and of the optimized version.*

We have chosen to measure performance through the total number of edges created by the programs, since the running time for the triangulation is proportional to this quantity. The difference between the running times of both versions becomes perceptible from 60 sites on, and, for 130,000 sites, the optimized version is already twice as fast as the standard one. The difference increases with n.

On a 200 Mhz Indy workstation, 200,000 sites are triangulated in 7 seconds, after a sorting phase of 3 seconds. The rate of triangulation (exclusive of sorting) is about 30,000 sites per second (between 50,000 and 60,000 triangles).

5 Conclusion

The evaluation of the expected number of unfinished sites in a k-rectangle, under quasi-uniform distribution, is an interesting result in itself, that may be used to analyze various algorithms. It may be added to the probabilistic results already obtained by Bern, Eppstein and Yao ([3]) on the Delaunay triangulation in any dimension.

However, the major interest of this result lies in its application to the construction of the Delaunay triangulation in any dimension with a divide-and-conquer algorithm based upon a balanced kd-tree partition scheme. This paper has shown the efficiency of such a scheme in the planar case. The corresponding algorithm is fairly easy to implement, and seems to be the fastest, at least for quasi-uniform distributions, to our knowledge.

One of our next research goals is to prove Theorem 3.1 for other classical distributions. Doing this would give our method a neat advantage over all those based on regular grids.

Another goal is to prove that the method may be applied in k-dimensional setting, to obtain, after multi-dimensional pre-sorting, a triangulation the expected running time of which would be proportional to the number of sites. It would be interesting to compare this method with Dwyer's ([4]), in which it is suggested to use a pre-partition of the sites based on a regular grid. In $3D$, the algorithm is almost established.

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